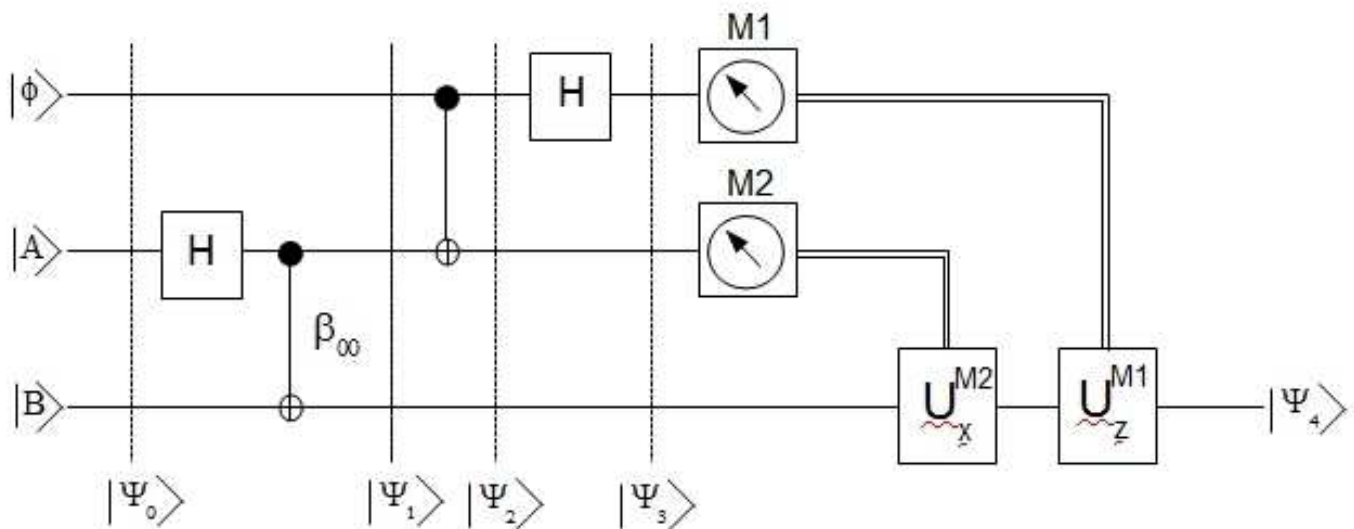> *restart*;

> *interface*(*warnlevel* = 0) :   # **Maple 12**

> *with*(*LinearAlgebra*) :

> *with*(*Bits*) :

This worksheet shows the matrices involved in the teleportation discussion of Chapter 5 - Quantum Teleportation

> *TP* :=**proc**(*M1*, *M2*) **return** *KroneckerProduct*(*M1*, *M2*) **end proc**:

> *T* :=**proc**(*x*) **return** *Transpose*(*x*) **end proc**:

> *VSte* := **proc**(*n*)   # *Generates a list of computational states for n qubits*
         local *i*, *L*;   # *e.g. n= 2 ⇒ $\left[\; |00\rangle \; |01\rangle \; |10\rangle \; |11\rangle \; \right]$*
         $L := Matrix(1, 2^n)$;
         Settings(*defaultbits* = *n*);
         for *i* **from** 1 **to** 2^n **do**
             $L[1, i] := cat(\grave{}/\grave{}, String(i-1, msbfirst), ")")$;
         end do;
      #   *print*(*L*);
         return *L*;   # *returns Matrix L*
      end proc:

Defining working matrices

> *I2* := *IdentityMatrix*(2);

$$I2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(1)

> I4 := *IdentityMatrix*(4);

$$I4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(2)

> CNOT := *RowOperation*(I4, [3, 4]);

$$CNOT := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(3)

> X := *RowOperation*(I2, [1, 2]);

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(4)

> Z := *Matrix*([[1, 0], [0, -1]]);

$$Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(5)

> H := $\dfrac{X + Z}{\sqrt{2}}$;

$$H := \begin{bmatrix} \dfrac{1}{2}\sqrt{2} & \dfrac{1}{2}\sqrt{2} \\ \dfrac{1}{2}\sqrt{2} & -\dfrac{1}{2}\sqrt{2} \end{bmatrix}$$

(6)

> GBell := *Multiply*(CNOT, TP(H, I2));

$$GBell := \begin{bmatrix} \dfrac{1}{2}\sqrt{2} & 0 & \dfrac{1}{2}\sqrt{2} & 0 \\ 0 & \dfrac{1}{2}\sqrt{2} & 0 & \dfrac{1}{2}\sqrt{2} \\ 0 & \dfrac{1}{2}\sqrt{2} & 0 & -\dfrac{1}{2}\sqrt{2} \\ \dfrac{1}{2}\sqrt{2} & 0 & -\dfrac{1}{2}\sqrt{2} & 0 \end{bmatrix}$$

(7)

> IBell := *MatrixInverse*(𝔾Bell);

$$IBell := \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & 0 & \frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ \frac{1}{2}\sqrt{2} & 0 & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 \end{bmatrix}$$

(8)

Defining State Vectors

Unknown state vector; that is, a_0 and a_1 are unknown. This is the quantum state to be " teleported"

$$|\phi\rangle = a_0|0\rangle + a_1|1\rangle$$

> Co := *Matrix*([[a[0]], [a[1]]]);
> St := *Transpose*(VSte(1)) :
> φ := *Multiply*(T(Co), St)[1, 1];

$$Co := \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$\varphi := a_0 \ |0\rangle + a_1 \ |1\rangle$$

(9)

- qubits used to generate the entangled states (correlated states, EPR pair)

$$|A\rangle = |0\rangle, \quad |B\rangle = |0\rangle \text{ and } |A\rangle \otimes |B\rangle = |00\rangle$$

$$\mathbb{G}Bell(|A\rangle \otimes |B\rangle) = \beta_{00} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

> A := *Matrix*([[1], [0]]) :
> B := *Matrix*([[1], [0]]) :
> C[0, 0] := *Multiply*(𝔾Bell, TP(A, B));
> St := *Transpose*(VSte(2)) :
> β[0, 0] := *factor*(*Multiply*(T(C[0, 0]), St)[1, 1]); # **Bell State** β00

$$C_{0, 0} := \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ 0 \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$\beta_{0, 0} := \frac{1}{2}\sqrt{2}\ (|00\rangle + |11\rangle)$$

(10)

The state vector representing the three qubit state

$|\psi_0\rangle = |\phi\rangle \otimes |A\rangle \otimes |B\rangle = a_0|000\rangle + a_1|100\rangle$

> $n := 3:$
$Co0 := TP(Co, TP(A, B));$
$St := Transpose(VSte(3)) : `|\psi 0\rangle := Multiply(T(Co0), St)[1, 1];$

$$Co0 := \begin{bmatrix} a_0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$|\psi 0\rangle := a_0 \,|000\rangle + a_1 \,|100\rangle$ **(11)**

Applying the $\mathbb{G}$Bell operator to the last two qubits of $|\psi_0\rangle$

$|\psi_1\rangle = (I2 \otimes \mathbb{G}\text{Bell})|\psi_0\rangle = a_0|0\rangle \otimes \mathbb{G}\text{Bell}(|00\rangle) + a_1|1\rangle \otimes \mathbb{G}\text{Bell}(|00\rangle)$

$|\psi_1\rangle = a_0|0\rangle \otimes \beta_{00} + a_1|1\rangle \otimes \beta_{00}$

$|\psi_1\rangle = a_0|0\rangle \otimes (\dfrac{|00\rangle + |11\rangle}{\sqrt{2}}) + a_1|1\rangle \otimes (\dfrac{|00\rangle + |11\rangle}{\sqrt{2}})$

$|\psi_1\rangle = a_0 \dfrac{|000\rangle + |011\rangle}{\sqrt{2}} + a_1 \dfrac{|100\rangle + |111\rangle}{\sqrt{2}}$

> $Co1 := Multiply(TP(I2, \mathbb{G}\text{Bell}), Co0); |\psi 1\rangle := factor(Multiply(T(Co1), St)[1, 1]);$

$$Co1 := \begin{bmatrix} \frac{1}{2}\sqrt{2}\,a_0 \\ 0 \\ 0 \\ \frac{1}{2}\sqrt{2}\,a_0 \\ \frac{1}{2}\sqrt{2}\,a_1 \\ 0 \\ 0 \\ \frac{1}{2}\sqrt{2}\,a_1 \end{bmatrix}$$

$|\psi 1\rangle := \dfrac{1}{2}\sqrt{2}\,(a_0\,|000\rangle + a_0\,|011\rangle + a_1\,|100\rangle + a_1\,|111\rangle)$ **(12)**

Applying the CNOT operator to the first two qubits of $|\psi_1\rangle$

$|\psi_2\rangle = (\text{CNOT} \otimes I2)|\psi_1\rangle$

$|\psi_2\rangle = a_0 \dfrac{CNOT(|00\rangle) \otimes |0\rangle \; + \; CNOT(|01\rangle) \otimes |1\rangle}{\sqrt{2}} \; + \; a_1 \dfrac{CNOT(|10\rangle) \otimes |0\rangle \; + \; CNOT(|11\rangle) \otimes |1\rangle}{\sqrt{2}}$

$|\psi_2\rangle = a_0 \dfrac{|000\rangle \; + \; |011\rangle}{\sqrt{2}} \; + \; a_1 \dfrac{|110\rangle \; + \; |101\rangle}{\sqrt{2}}$

> $Co2 := Multiply(TP(CNOT, I2), Co1);$
> $|\psi2\rangle := factor(Multiply(T(Co2), St)[1, 1]);$

$$Co2 := \begin{bmatrix} \dfrac{1}{2}\sqrt{2}\,a_0 \\[4pt] 0 \\[4pt] 0 \\[4pt] \dfrac{1}{2}\sqrt{2}\,a_0 \\[4pt] 0 \\[4pt] \dfrac{1}{2}\sqrt{2}\,a_1 \\[4pt] \dfrac{1}{2}\sqrt{2}\,a_1 \\[4pt] 0 \end{bmatrix}$$

$$|\psi2\rangle := \frac{1}{2}\sqrt{2}\,\left(a_0\,|000\rangle + a_0\,|011\rangle + a_1\,|101\rangle + a_1\,|110\rangle\right)$$ (13)

Applying the Hadamard operator to the first qubit of $|\psi_2\rangle$

$|\psi_3\rangle = a_0 \dfrac{H|0\rangle \otimes (|00\rangle \; + \; |11\rangle)}{\sqrt{2}} \; + \; a_1 \dfrac{H|1\rangle \otimes (|10\rangle \; + \; |01\rangle)}{\sqrt{2}}$

$|\psi_3\rangle = a_0 \dfrac{(|0\rangle + |1\rangle) \otimes (|00\rangle \; + \; |11\rangle)}{2} \; + \; a_1 \dfrac{(|0\rangle - |1\rangle) \otimes (|10\rangle \; + \; |01\rangle)}{2}$

$|\psi_3\rangle = a_0 \dfrac{(|000\rangle \; + \; |011\rangle + |100\rangle \; + \; |111\rangle)}{2} \; + \; a_1 \dfrac{(|010\rangle \; + \; |001\rangle - |110\rangle \; - \; |101\rangle)}{2}$

> $Co3 := Multiply(TP(\text{H}, \text{I4}), Co2);$
$|\psi3\rangle := Multiply(T(Co3), St)[1, 1];$

$$Co3 := \begin{bmatrix} \dfrac{1}{2}\, a_0 \\[2mm] \dfrac{1}{2}\, a_1 \\[2mm] \dfrac{1}{2}\, a_1 \\[2mm] \dfrac{1}{2}\, a_0 \\[2mm] \dfrac{1}{2}\, a_0 \\[2mm] -\dfrac{1}{2}\, a_1 \\[2mm] -\dfrac{1}{2}\, a_1 \\[2mm] \dfrac{1}{2}\, a_0 \end{bmatrix}$$

$$|\psi3\rangle := \frac{1}{2}\, a_0\, |000\rangle + \frac{1}{2}\, a_1\, |001\rangle + \frac{1}{2}\, a_1\, |010\rangle + \frac{1}{2}\, a_0\, |011\rangle + \frac{1}{2}\, a_0\, |100\rangle - \frac{1}{2}\, a_1\, |101\rangle \tag{14}$$
$$- \frac{1}{2}\, a_1\, |110\rangle + \frac{1}{2}\, a_0\, |111\rangle$$

Another way of obtaining $|\psi_3\rangle$ is by applying the inverse Bell Gate operator IBell to the first two qubits of state $|\psi_1\rangle$. Thus $|\psi_1\rangle \rightarrow |\psi_3\rangle$,

$$|\psi_3\rangle = (\text{IBell} \otimes \text{I2})|\psi_1\rangle$$
$$|\psi_3\rangle = a_0\, \frac{\text{IBell}(|00\rangle) \otimes |0\rangle \; + \; \text{IBell}(|01\rangle) \otimes |1\rangle}{\sqrt{2}} + a_1\, \frac{\text{IBell}(|10\rangle) \otimes |0\rangle \; + \; \text{IBell}(|11\rangle) \otimes |1\rangle}{\sqrt{2}}$$

> $Co3 := Multiply(TP(\text{IBell}, I2), Co1);$
  $`|\psi 3\rangle := Multiply(T(Co3), St)[1, 1];$

$$Co3 := \begin{bmatrix} \dfrac{1}{2}\,a_0 \\[2mm] \dfrac{1}{2}\,a_1 \\[2mm] \dfrac{1}{2}\,a_1 \\[2mm] \dfrac{1}{2}\,a_0 \\[2mm] \dfrac{1}{2}\,a_0 \\[2mm] -\dfrac{1}{2}\,a_1 \\[2mm] -\dfrac{1}{2}\,a_1 \\[2mm] \dfrac{1}{2}\,a_0 \end{bmatrix}$$

$$|\psi 3\rangle := \frac{1}{2}\,a_0\,|000\rangle + \frac{1}{2}\,a_1\,|001\rangle + \frac{1}{2}\,a_1\,|010\rangle + \frac{1}{2}\,a_0\,|011\rangle + \frac{1}{2}\,a_0\,|100\rangle - \frac{1}{2}\,a_1\,|101\rangle \qquad\qquad \textbf{(15)}$$

$$- \frac{1}{2}\,a_1\,|110\rangle + \frac{1}{2}\,a_0\,|111\rangle$$

Now we write the density matrix of $|\psi_3\rangle$.

$$\rho = |\psi_3\rangle\langle\psi_3|$$

> $\rho := TP(Co3, HermitianTranspose(Co3));$ # note $\overline{a_0}$ refers to the complex conjugate of a_0

$$\rho := \begin{bmatrix}
\frac{1}{4}a_0\overline{a_0} & \frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_0} & \frac{1}{4}a_0\overline{a_0} & -\frac{1}{4}a_0\overline{a_1} & -\frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_0} \\[4pt]
\frac{1}{4}a_1\overline{a_0} & \frac{1}{4}a_1\overline{a_1} & \frac{1}{4}a_1\overline{a_1} & \frac{1}{4}a_1\overline{a_0} & \frac{1}{4}a_1\overline{a_0} & -\frac{1}{4}a_1\overline{a_1} & -\frac{1}{4}a_1\overline{a_1} & \frac{1}{4}a_1\overline{a_0} \\[4pt]
\frac{1}{4}a_1\overline{a_0} & \frac{1}{4}a_1\overline{a_1} & \frac{1}{4}a_1\overline{a_1} & \frac{1}{4}a_1\overline{a_0} & \frac{1}{4}a_1\overline{a_0} & -\frac{1}{4}a_1\overline{a_1} & -\frac{1}{4}a_1\overline{a_1} & \frac{1}{4}a_1\overline{a_0} \\[4pt]
\frac{1}{4}a_0\overline{a_0} & \frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_0} & \frac{1}{4}a_0\overline{a_0} & -\frac{1}{4}a_0\overline{a_1} & -\frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_0} \\[4pt]
\frac{1}{4}a_0\overline{a_0} & \frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_0} & \frac{1}{4}a_0\overline{a_0} & -\frac{1}{4}a_0\overline{a_1} & -\frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_0} \\[4pt]
-\frac{1}{4}a_1\overline{a_0} & -\frac{1}{4}a_1\overline{a_1} & -\frac{1}{4}a_1\overline{a_1} & -\frac{1}{4}a_1\overline{a_0} & -\frac{1}{4}a_1\overline{a_0} & \frac{1}{4}a_1\overline{a_1} & \frac{1}{4}a_1\overline{a_1} & -\frac{1}{4}a_1\overline{a_0} \\[4pt]
-\frac{1}{4}a_1\overline{a_0} & -\frac{1}{4}a_1\overline{a_1} & -\frac{1}{4}a_1\overline{a_1} & -\frac{1}{4}a_1\overline{a_0} & -\frac{1}{4}a_1\overline{a_0} & \frac{1}{4}a_1\overline{a_1} & \frac{1}{4}a_1\overline{a_1} & -\frac{1}{4}a_1\overline{a_0} \\[4pt]
\frac{1}{4}a_0\overline{a_0} & \frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_0} & \frac{1}{4}a_0\overline{a_0} & -\frac{1}{4}a_0\overline{a_1} & -\frac{1}{4}a_0\overline{a_1} & \frac{1}{4}a_0\overline{a_0}
\end{bmatrix}$$

(16)