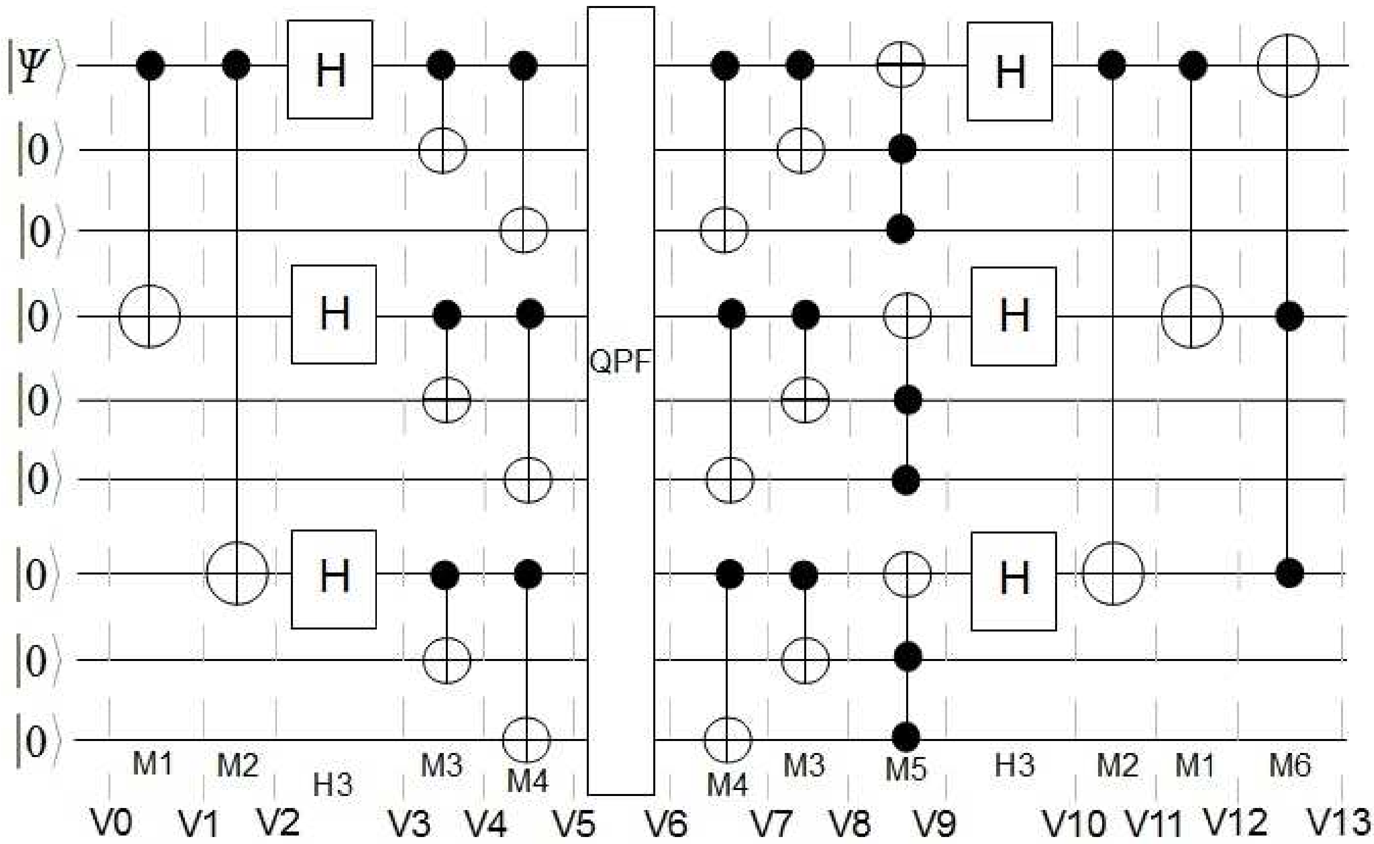
```
> restart :
> interface(warnlevel = 0) :          # Maple 12
> with(LinearAlgebra) :
> with(Bits) :
> Settings(defaultbits = 9) :
```



Error Selection

qp = 1 : phase-flip and qubit-flip at 1st qubit
= 2 : phase-flip and qubit-flip at 2
= 3 : phase-flip and qubit-flip at 3
= 4 : no phase flips, qubit-flip at 1
= 5 : no phase-flips, qubit-flip at 2
= 6 : no phase-flips, qubit-flip at 3
= 7 : phase-flip at 1, no qubit-flips
= 8 : phase-flip at 4, no qubit-flips
= 9 : phase-flip at 7, no qubit-flips
= 10 : phase-flip & qubit-flip at 6

```
> qp := 1 :
```

Utility functions

```
> K := proc(a, b) return KroneckerProduct(a, b) end proc:
> T := proc(x) return Transpose(x) end proc:
> VSte := proc(n)                # Generates a list of  computational states for  n qubits
          local i, L;            #  e.g.  n= 2  ⇒ [ |00〉 |01〉 |10〉 |11〉 ]
          L := Matrix(1, 2^n);
          Settings(defaultbits = n);
          for i from 1 to 2^n do
              L[1, i] := cat(`|`, String(i − 1, msbfirst), "〉");
          end do;
     #    print(L);
        return L;        # returns Matrix L
      end proc:

> Vr := proc(x, y)          # prints out the 512 by 1 matrix as (±α±β) ( representation of the state)
          local i, j, b, d, o;
   # example: α|1001〉 + β|1001〉 →(α + β) |009〉                 Vr(X, 1) dec
          for i from 1 to 512 do      #                        → (α + β) |009〉,|012〉         Vr(X,
   2) dec, oct
              if x[i, 1] ≠ 0 then     #                        → (α + β) |009〉,|012〉,|1001〉  Vr(X,
   2) dec, oct, bin
                  j := i − 1;
                  if  j ≤ 10 then
                      d := cat(`|00`, j, "〉");
                  elif j ≤ 99 then
                      d := cat(`|0`, j, "〉");
                  else
                      d := cat(`|`, j, "〉");
                  end if;
                  if  j ≤ 7 then
                      o := cat(`|00`, convert( j, octal), "〉");
                  elif j ≤ 63 then                       # j ≤77_o
                      o := cat(`|0`, convert( j, octal), "〉");
                  else
                      o := cat(`|`, convert( j, octal), "〉");
                  end if;
                  b := cat(`|`, String( j, msbfirst), "〉");
                 if  y = 1 then print( (x[i, 1]) [d])
                     elif  y = 2 then  print( (x[i, 1]) [d, o])
                     elif  y = 3 then print( (x[i, 1]) [d, o, b]);
                  end if;
              end;
          end do;
         end proc:
```

```
> Gxt := proc(bit, h)                    # generates Toffoli type matrices such as M6
        local i, j, a, b, c, d, e, s;
        global M;
        a := bit - h;
        b := 2^(a + 1);
        c := (2^(bit - 1) - 2^a) + 2;
        s := 2^a + 2;
        d := b - s;
        e := 2^(bit - 1);
        for i from s to c by b do
            for j from i to i + d by 2 do
                M := RowOperation(M, [j, (j + e)])
            end do;
        end do;
    end proc:
```

```
> Gxnot := proc(bit)                     # generates xnot type matrices such as M1 and M2
        local i, srow, lrow;
        global M;
        srow := 2^(bit - 1) + 1;
        lrow := 2^bit - 1;
        for i from srow to lrow by 2 do
            M := RowOperation(M, [i, (i + 1)]);
        end do;
    end proc:
```

Utility matrices

```
> I2 := IdentityMatrix(2) :
> I4 := IdentityMatrix(4) :
> I8 := IdentityMatrix(8) :
> I16 := IdentityMatrix(16) :
> I32 := IdentityMatrix(32) :
> I64 := IdentityMatrix(64) :
> I128 := IdentityMatrix(128) :
```

Utility Operators

```
> Uz := RowOperation(I2, 2, -1);         # phase-flip operator
```

$$Uz := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \textbf{(1)}$$

```
> Ux := RowOperation(I2, [1, 2]);        # qubit-flip operator
```

$$Ux := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \textbf{(2)}$$

> $CNOT := RowOperation(I4, [3, 4]);$

$$CNOT := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (3)$$

> $G23 := RowOperation(I4, [2, 3]);$ # *qubit-exchange operator*

$$G23 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

> $H := \frac{1}{\sqrt{2}} Matrix([[1, 1], [1, -1]]);$

$$H := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix} \quad (5)$$

M1 - M6, H3, and QPF matrices/operators

> $\mathrm{M} := \mathrm{I16}:$
$\mathrm{M1} := \mathrm{K}(\mathrm{Gxnot}(4), \mathrm{I32});$

$$M1 := \begin{bmatrix} \textit{512 x 512 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \quad (6)$$

> $\mathrm{M} := \mathrm{I128}:$
$\mathrm{M2} := \mathrm{K}(\mathrm{Gxnot}(7), \mathrm{I4});$

$$M2 := \begin{bmatrix} \textit{512 x 512 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \quad (7)$$

> $H3 := K(K(H, I4), K(K(H, I4), K(H, I4))):$
$'\sqrt{8} \cdot H3' = \sqrt{8}\, H3;$

$$\sqrt{8}\ H3 = \begin{bmatrix} \textit{512 x 512 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \quad (8)$$

> $M3 := K(K(K(CNOT, I2), K(CNOT, I2)), K(CNOT, I2));$

$$M3 := \begin{bmatrix} \textit{512 x 512 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \tag{9}$$

> $M4 := K(Multiply(K(G23, I2), Multiply(K(I2, CNOT), K(G23, I2))),$
> $K(Multiply(K(G23, I2), Multiply(K(I2, CNOT), K(G23, I2))),$
> $Multiply(K(G23, I2), Multiply(K(I2, CNOT), K(G23, I2)))));$

$$M4 := \begin{bmatrix} \textit{512 x 512 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \tag{10}$$

IT matrix/operator is a variation of the Toffoli gate. The 2nd and 3rd qubits are the control qubits. When the state of both qubits equal "1" the 1st qubit is "flipped"

$$c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + \mathbf{c_3|011\rangle} + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + \mathbf{c_7|111\rangle}$$

$$c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + \mathbf{c_3|111\rangle} + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + \mathbf{c_7|011\rangle}$$

> $IT := RowOperation(I8, [8, 4]):$
> M5 := K(K(IT, IT), IT);

$$M5 := \begin{bmatrix} \textit{512 x 512 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \tag{11}$$

> $M := I128:$
> $M6 := K(Gxt(7, 4), I4);$

$$M6 := \begin{bmatrix} \textit{512 x 512 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \tag{12}$$

QPF: qubit error operator

```
> if qp = 1 then
   QPF := K(Multiply(Ux, Uz), IdentityMatrix(2^8)); # phase-flip & qubit-flip at 1
  elif qp = 2 then
     QPF := K(I2, K(Multiply(Ux, Uz), IdentityMatrix(2^7))); # phase-flip  & qubit-flip at 2
  elif qp = 3 then
     QPF := K(I4, K(Multiply(Ux, Uz), IdentityMatrix(2^6))); # phase-flip  & qubit-flip at 3
  elif qp = 4 then
     QPF := K(Multiply(Ux, I2), IdentityMatrix(2^8)); # no phase-flip & qubit-flip at 1
  elif qp = 5 then
     QPF := K(I2, K(Multiply(Ux, I2), IdentityMatrix(2^7))); # no phase-flip & qubit-flip at 2
  elif qp = 6 then
     QPF := K(I4, K(Multiply(Ux, I2), IdentityMatrix(2^6))); # no phase-flip & qubit-flip at 3
  elif qp = 7 then
     QPF := K(Multiply(I2, Uz), IdentityMatrix(2^8)); #  phase-flip at 1 &  no qubit-flip
  elif qp = 8 then
     QPF := K(I8, K(Multiply(I2, Uz), IdentityMatrix(2^5))); #  phase-flip at 4 & no qubit-flip
  elif qp = 9 then
     QPF := K(I64, K(Multiply(I2, Uz), IdentityMatrix(2^2))); #  phase-flip at 7 & no qubit-flip
  elif qp = 10 then
     QPF := K(I32, K(Multiply(Ux, Uz), IdentityMatrix(2^3))); # phase-flip & qubit-flip at 6
  end if;
```

$$QPF := \left[\begin{array}{c} \text{512 x 512 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{array}\right] \tag{13}$$

$V0 = (\alpha|0\rangle + \beta|1\rangle) \otimes |00000000\rangle = \alpha|000000000\rangle + \beta|100000000\rangle$

The octal representation: $\alpha|000\rangle_o + \beta|400\rangle_o$

```
> n := 9 :
  q1 := Matrix([[α], [β]]) :    #  α|0⟩ + β|1⟩
  qn := Matrix([[1], [0]]) :     #    |0⟩
  Co0 := K(q1, K(qn, K(qn, K(qn, K(qn, K(qn, K(qn, K(qn, qn))))))));
  Vr(Co0, 3); St := Transpose(VSte(n)) :
  |V0⟩ := Multiply(T(Co0), St)[1, 1];
```

$$Co0 := \left[\begin{array}{c} \text{512 x 1 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{array}\right]$$

$$\alpha_{|000\rangle, |000\rangle, |000000000\rangle}$$

$$\beta_{|256\rangle, |400\rangle, |100000000\rangle}$$

$$|V0\rangle := \alpha\,|000000000\rangle + \beta\,|100000000\rangle \tag{14}$$

$$\mathbf{V1 = M1 \cdot V0}$$
$$= \mathbf{M1(\alpha|000000000\rangle + \beta|100000000\rangle)}$$
$$= \mathbf{\alpha|000000000\rangle + \beta|100100000\rangle}$$
$$= \mathbf{\alpha|000\rangle_o + \beta|440\rangle_o}$$

```
> Co1 := Multiply(M1, Co0);
 Vr(Co1, 3);
|V1⟩ := Multiply(T(Co1), St)[1, 1];
```

$$Co1 := \begin{bmatrix} \text{512 x 1 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix}$$

$$\alpha_{|000\rangle, |000\rangle, |000000000\rangle}$$

$$\beta_{|288\rangle, |440\rangle, |100100000\rangle}$$

$$|V1\rangle := \alpha\, |000000000\rangle + \beta\, |100100000\rangle \tag{15}$$

$$\mathbf{V2 = M2 \cdot V1}$$
$$= \mathbf{M2(\alpha|000000000\rangle + \beta|100100000\rangle)}$$
$$= \mathbf{\alpha|000000000\rangle + \beta|100100100\rangle}$$
$$= \mathbf{\alpha|000\rangle_o + \beta|444\rangle_o}$$

```
> Co2 := Multiply(M2, Co1);
 Vr(Co2, 3);
|V2⟩ := Multiply(T(Co2), St)[1, 1];
```

$$Co2 := \begin{bmatrix} \text{512 x 1 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix}$$

$$\alpha_{|000\rangle, |000\rangle, |000000000\rangle}$$

$$\beta_{|292\rangle, |444\rangle, |100100100\rangle}$$

$$|V2\rangle := \alpha\, |000000000\rangle + \beta\, |100100100\rangle \tag{16}$$

The Hadamard H3 matrix/operator

$$H\,|0\rangle = \frac{\alpha|0\rangle + \beta|1\rangle}{\sqrt{2}};\;\; H\,|1\rangle = \frac{\alpha|0\rangle - \beta|1\rangle}{\sqrt{2}},$$

$$\begin{aligned} \mathbf{V3} = {} & \alpha\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle \\ & + \beta\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle \end{aligned}$$

$$\begin{aligned} \mathbf{V3} = {} & (\alpha + \beta)|000000000\rangle + (\alpha - \beta)|000000100\rangle + (\alpha - \beta)|000100000\rangle + (\alpha + \beta)|000100100\rangle \\ & + (\alpha - \beta)|100000000\rangle + (\alpha + \beta)|100000100\rangle + (\alpha + \beta)|100100000\rangle + (\alpha - \beta)|100100100\rangle \end{aligned}$$

$$\begin{aligned} \mathbf{V3} = {} & (\alpha + \beta)|000\rangle_o + (\alpha - \beta)|004\rangle_o + (\alpha - \beta)|040\rangle_o + (\alpha + \beta)|044\rangle_o \\ & + (\alpha - \beta)|400\rangle_o + (\alpha + \beta)|404\rangle_o + (\alpha + \beta)|440\rangle_o + (\alpha - \beta)|444\rangle_o \end{aligned}$$

$$\begin{aligned} \mathbf{V3} = {} & (\alpha + \beta)|0\rangle + (\alpha - \beta)|4\rangle + (\alpha - \beta)|32\rangle + (\alpha + \beta)|36\rangle \\ & + (\alpha - \beta)|256\rangle + (\alpha + \beta)|260\rangle + (\alpha + \beta)|288\rangle + (\alpha - \beta)|292\rangle \end{aligned}$$

```
> Co3 := simplify(sqrt(8) Multiply(H3, Co2));
  Vr(Co3, 3);
  |V3> := Multiply(T(Co3), St)[1, 1];
```

$$Co3 := \left[\begin{array}{c} \mathit{512\ x\ 1\ Matrix} \\ \mathit{Data\ Type:\ anything} \\ \mathit{Storage:\ rectangular} \\ \mathit{Order:\ Fortran\_order} \end{array}\right]$$

$$(\alpha+\beta)_{|000\rangle,\ |000\rangle,\ |000000000\rangle}$$
$$(\alpha-\beta)_{|004\rangle,\ |004\rangle,\ |000000100\rangle}$$
$$(\alpha-\beta)_{|032\rangle,\ |040\rangle,\ |000100000\rangle}$$
$$(\alpha+\beta)_{|036\rangle,\ |044\rangle,\ |000100100\rangle}$$
$$(\alpha-\beta)_{|256\rangle,\ |400\rangle,\ |100000000\rangle}$$
$$(\alpha+\beta)_{|260\rangle,\ |404\rangle,\ |100000100\rangle}$$
$$(\alpha+\beta)_{|288\rangle,\ |440\rangle,\ |100100000\rangle}$$
$$(\alpha-\beta)_{|292\rangle,\ |444\rangle,\ |100100100\rangle}$$

$$\begin{aligned} |V3\rangle := {} & (\alpha+\beta)\,|000100100\rangle + (\alpha-\beta)\,|100000000\rangle + (\alpha+\beta)\,|100000100\rangle + (\alpha+\beta)\,|100100000\rangle \\ & + (\alpha-\beta)\,|100100100\rangle + (\alpha-\beta)\,|000100000\rangle + (\alpha-\beta)\,|000000100\rangle + (\alpha+\beta)\,|000000000\rangle \end{aligned} \quad \mathbf{(17)}$$

$$\mathbf{V4 = M3 \cdot V3}$$
$$\mathbf{= (\alpha + \beta)|000\rangle_o + (\alpha - \beta)|006\rangle_o + (\alpha - \beta)|060\rangle_o + (\alpha + \beta)|066\rangle_o}$$
$$\mathbf{+ (\alpha - \beta)|600\rangle_o + (\alpha + \beta)|606\rangle_o + (\alpha + \beta)|660\rangle_o + (\alpha - \beta)|666\rangle_o}$$

```
> Co4 := Multiply(M3, Co3);
  Vr(Co4, 3) :
  `|V4⟩` := Multiply(T(Co4), St)[1, 1];
```

$$(\alpha+\beta)_{|000\rangle, |000\rangle, |000000000\rangle}$$
$$(\alpha-\beta)_{|006\rangle, |006\rangle, |000000110\rangle}$$
$$(\alpha-\beta)_{|048\rangle, |060\rangle, |000110000\rangle}$$
$$(\alpha+\beta)_{|054\rangle, |066\rangle, |000110110\rangle}$$
$$(\alpha-\beta)_{|384\rangle, |600\rangle, |110000000\rangle}$$
$$(\alpha+\beta)_{|390\rangle, |606\rangle, |110000110\rangle}$$
$$(\alpha+\beta)_{|432\rangle, |660\rangle, |110110000\rangle}$$
$$(\alpha-\beta)_{|438\rangle, |666\rangle, |110110110\rangle}$$

$$|V4\rangle := (\alpha+\beta)\,|110000110\rangle + (\alpha+\beta)\,|110110000\rangle + (\alpha-\beta)\,|110110110\rangle + (\alpha-\beta)\,|000000110\rangle + (\alpha-\beta)\,|000110000\rangle + (\alpha+\beta)\,|000110110\rangle + (\alpha-\beta)\,|110000000\rangle + (\alpha+\beta)\,|000000000\rangle \quad \mathbf{(18)}$$

$$\mathbf{V5 = M4 \cdot V4}$$
$$\mathbf{= (\alpha + \beta)|000\rangle_o + (\alpha - \beta)|007\rangle_o + (\alpha - \beta)|070\rangle_o + (\alpha + \beta)|077\rangle_o}$$
$$\mathbf{+ (\alpha - \beta)|700\rangle_o + (\alpha + \beta)|707\rangle_o + (\alpha + \beta)|770\rangle_o + (\alpha - \beta)|777\rangle_o}$$

```
> Co5 := Multiply(M4, Co4);
  Vr(Co5, 3) :
  |V5⟩ := Multiply(T(Co5), St)[1, 1];
```

$$(\alpha+\beta)_{|000\rangle, |000\rangle, |000000000\rangle}$$
$$(\alpha-\beta)_{|007\rangle, |007\rangle, |000000111\rangle}$$
$$(\alpha-\beta)_{|056\rangle, |070\rangle, |000111000\rangle}$$
$$(\alpha+\beta)_{|063\rangle, |077\rangle, |000111111\rangle}$$
$$(\alpha-\beta)_{|448\rangle, |700\rangle, |111000000\rangle}$$
$$(\alpha+\beta)_{|455\rangle, |707\rangle, |111000111\rangle}$$
$$(\alpha+\beta)_{|504\rangle, |770\rangle, |111111000\rangle}$$
$$(\alpha-\beta)_{|511\rangle, |777\rangle, |111111111\rangle}$$

$$|V5\rangle := (\alpha-\beta)\,|000000111\rangle + (\alpha-\beta)\,|000111000\rangle + (\alpha+\beta)\,|000111111\rangle + (\alpha-\beta)\,|111000000\rangle + (\alpha+\beta)\,|111000111\rangle + (\alpha+\beta)\,|111111000\rangle + (\alpha-\beta)\,|111111111\rangle + (\alpha+\beta)\,|000000000\rangle \quad \mathbf{(19)}$$

Qubit Error: QPF

A phase-flip error on the first qubit

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle - \beta|1\rangle$$

or

$$\alpha|+\rangle + \beta|-\rangle \rightarrow \alpha|-\rangle + \beta|+\rangle$$

A qubit-flip error on the first qubit

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|1\rangle + \beta|0\rangle$$

Both types of errors

$$\alpha\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right) \rightarrow \alpha\left(\frac{|100\rangle-|011\rangle}{\sqrt{2}}\right)$$

$$\beta\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right) \rightarrow \beta\left(\frac{|100\rangle+|011\rangle}{\sqrt{2}}\right)$$

if qp = 1

$$V6 = QPF \cdot V7$$

$$= (\alpha + \beta)|400\rangle_o + (\alpha - \beta)|407\rangle_o + (\alpha - \beta)|470\rangle_o + (\alpha + \beta)|477\rangle_o$$

$$+ (-\alpha + \beta)|300\rangle_o + (-\alpha - \beta)|307\rangle_o - (\alpha - \beta)|370\rangle_o + (-\alpha + \beta)|377\rangle_o$$

```
> Co6 := Multiply(QPF, Co5);
  Vr(Co6, 3);
  |V6⟩ := Multiply(T(Co6), St)[1, 1];
```

$$Co6 := \begin{bmatrix} \textit{512 x 1 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix}$$

$$(-\alpha + \beta)_{|192\rangle, |300\rangle, |011000000\rangle}$$

$$(-\alpha - \beta)_{|199\rangle, |307\rangle, |011000111\rangle}$$

$$(-\alpha - \beta)_{|248\rangle, |370\rangle, |011111000\rangle}$$

$$(-\alpha + \beta)_{|255\rangle, |377\rangle, |011111111\rangle}$$

$$(\alpha + \beta)_{|256\rangle, |400\rangle, |100000000\rangle}$$

$$(\alpha - \beta)_{|263\rangle, |407\rangle, |100000111\rangle}$$

$$(\alpha - \beta)_{|312\rangle, |470\rangle, |100111000\rangle}$$

$$(\alpha + \beta)_{|319\rangle, |477\rangle, |100111111\rangle}$$

$$|V6\rangle := (-\alpha + \beta)\ |011000000\rangle + (-\alpha - \beta)\ |011000111\rangle + (-\alpha - \beta)\ |011111000\rangle + (-\alpha + \beta)\ |011111111\rangle + (\alpha + \beta)\ |100000000\rangle + (\alpha - \beta)\ |100000111\rangle + (\alpha - \beta)\ |100111000\rangle + (\alpha + \beta)\ |100111111\rangle \quad \textbf{(20)}$$

if qp = 1

V7 = M4 · V6

$= (\alpha + \beta)|500\rangle_o + (\alpha - \beta)|506\rangle_o + (\alpha - \beta)|560\rangle_o + (\alpha + \beta)|566\rangle_o$

$+ (-\alpha + \beta)|300\rangle_o + (-\alpha - \beta)|306\rangle_o + (-\alpha - \beta)|360\rangle_o + (-\alpha + \beta)|366\rangle_o$

```
> Co7 := Multiply(M4, Co6);
  Vr(Co7, 3) : |V7⟩ := Multiply(T(Co7), St)[1, 1];
```

$$(-\alpha+\beta)_{|192\rangle, |300\rangle, |011000000\rangle}$$
$$(-\alpha-\beta)_{|198\rangle, |306\rangle, |011000110\rangle}$$
$$(-\alpha-\beta)_{|240\rangle, |360\rangle, |011110000\rangle}$$
$$(-\alpha+\beta)_{|246\rangle, |366\rangle, |011110110\rangle}$$
$$(\alpha+\beta)_{|320\rangle, |500\rangle, |101000000\rangle}$$
$$(\alpha-\beta)_{|326\rangle, |506\rangle, |101000110\rangle}$$
$$(\alpha-\beta)_{|368\rangle, |560\rangle, |101110000\rangle}$$
$$(\alpha+\beta)_{|374\rangle, |566\rangle, |101110110\rangle}$$

$$|V7\rangle := (-\alpha-\beta)\,|011000110\rangle + (-\alpha-\beta)\,|011110000\rangle + (-\alpha+\beta)\,|011110110\rangle + (\alpha+\beta)\,|101000000\rangle + (-\alpha+\beta)\,|011000000\rangle + (\alpha-\beta)\,|101000110\rangle + (\alpha-\beta)\,|101110000\rangle + (\alpha+\beta)\,|101110110\rangle \quad \textbf{(21)}$$

if qp = 1

V8 = M3 · V7

$= (\alpha + \beta)|700\rangle_o + (\alpha - \beta)|704\rangle_o + (\alpha - \beta)|740\rangle_o + (\alpha + \beta)|744\rangle_o$

$+ (-\alpha + \beta)|300\rangle_o + (-\alpha - \beta)|304\rangle_o + (-\alpha - \beta)|340\rangle_o + (\alpha - \beta)|344\rangle_o$

```
> Co8 := Multiply(M3, Co7);
  Vr(Co8, 3) : |V8⟩ := Multiply(T(Co8), St)[1, 1];
```

$$(-\alpha+\beta)_{|192\rangle, |300\rangle, |011000000\rangle}$$
$$(-\alpha-\beta)_{|196\rangle, |304\rangle, |011000100\rangle}$$
$$(-\alpha-\beta)_{|224\rangle, |340\rangle, |011100000\rangle}$$
$$(-\alpha+\beta)_{|228\rangle, |344\rangle, |011100100\rangle}$$
$$(\alpha+\beta)_{|448\rangle, |700\rangle, |111000000\rangle}$$
$$(\alpha-\beta)_{|452\rangle, |704\rangle, |111000100\rangle}$$
$$(\alpha-\beta)_{|480\rangle, |740\rangle, |111100000\rangle}$$
$$(\alpha+\beta)_{|484\rangle, |744\rangle, |111100100\rangle}$$

$$|V8\rangle := (-\alpha-\beta)\,|011100000\rangle + (-\alpha+\beta)\,|011100100\rangle + (\alpha+\beta)\,|111000000\rangle + (\alpha-\beta)\,|111000100\rangle + (-\alpha+\beta)\,|011000000\rangle + (-\alpha-\beta)\,|011000100\rangle + (\alpha-\beta)\,|111100000\rangle + (\alpha+\beta)\,|111100100\rangle \quad \textbf{(22)}$$

if qp = 1

V10 = M5 · V9

$= (\alpha + \beta)|300\rangle_o + (\alpha - \beta)|304\rangle_o + (\alpha - \beta)|340\rangle_o + (\alpha + \beta)|344\rangle_o$

$+ (-\alpha + \beta)|700\rangle_o + (-\alpha - \beta)|704\rangle_o + (-\alpha - \beta)|740\rangle_o + (-\alpha + \beta)|744\rangle_o$

```
> Co9 := Multiply(M5, Co8);
  Vr(Co9, 3) : `|V9⟩` := Multiply(T(Co9), St)[1, 1];
```

$$(\alpha+\beta)_{|192\rangle, |300\rangle, |011000000\rangle}$$
$$(\alpha-\beta)_{|196\rangle, |304\rangle, |011000100\rangle}$$
$$(\alpha-\beta)_{|224\rangle, |340\rangle, |011100000\rangle}$$
$$(\alpha+\beta)_{|228\rangle, |344\rangle, |011100100\rangle}$$
$$(-\alpha+\beta)_{|448\rangle, |700\rangle, |111000000\rangle}$$
$$(-\alpha-\beta)_{|452\rangle, |704\rangle, |111000100\rangle}$$
$$(-\alpha-\beta)_{|480\rangle, |740\rangle, |111100000\rangle}$$
$$(-\alpha+\beta)_{|484\rangle, |744\rangle, |111100100\rangle}$$

$$|V9\rangle := (\alpha+\beta)\,|011000000\rangle + (\alpha-\beta)\,|011000100\rangle + (\alpha-\beta)\,|011100000\rangle + (\alpha+\beta)\,|011100100\rangle + (-\alpha+\beta)\,|111000000\rangle + (-\alpha-\beta)\,|111000100\rangle + (-\alpha-\beta)\,|111100000\rangle + (-\alpha+\beta)\,|111100100\rangle \quad (23)$$

When qb = 1: phase-flip and qubit-flip errors at q1

$$V9 = \alpha\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\left|11\right\rangle\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle$$
$$+\ \beta\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left|11\right\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle$$

$$= \alpha(q1 \otimes|11\rangle\otimes q4 \otimes|00\rangle\otimes q7 \otimes|00\rangle) + \beta(q1 \otimes|11\rangle\otimes q4 \otimes|00\rangle\otimes q7 \otimes|00\rangle)$$

Apply the H3 operator

$$V10 = \alpha(|1\rangle\otimes|11\rangle\otimes |0\rangle\otimes|00\rangle\otimes |0\rangle\otimes|00\rangle) + \beta(|0\rangle\otimes|11\rangle\otimes|1\rangle \otimes|00\rangle\otimes |1\rangle\otimes|00\rangle)$$
$$= \alpha|111000000\rangle + \beta|011100100\rangle$$
$$= \alpha|700\rangle_o + \beta|344\rangle_o$$

```
> Co10 := simplify( 1/sqrt(8) Multiply(H3, Co9) );
  Vr(Co10, 3) : |V10⟩ := Multiply(T(Co10), St)[1, 1];
```

$$\beta_{|228\rangle, |344\rangle, |011100100\rangle}$$
$$\alpha_{|448\rangle, |700\rangle, |111000000\rangle}$$

$$|V10\rangle := \beta\,|011100100\rangle + \alpha\,|111000000\rangle \quad (24)$$

if qp = 1

V11 = M2 · V10

= M2(α|111000000⟩+β|011100100⟩

= α|111000100⟩+β|011100100⟩

= $\alpha|704\rangle_o + \beta|344\rangle_o$

> $Co11 := Multiply(M2, Co10);$
> $Vr(Co11, 3);$ `$|V11\rangle$` $:= Multiply(T(Co11), St)[1, 1];$

$$Co11 := \begin{bmatrix} \textit{512 x 1 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix}$$

$$\beta_{|228\rangle, |344\rangle, |011100100\rangle}$$

$$\alpha_{|452\rangle, |704\rangle, |111000100\rangle}$$

$$|V11\rangle := \alpha\,|111000100\rangle + \beta\,|011100100\rangle \quad \textbf{(25)}$$

if qp = 1

V12 = M1 · V11

= M1(α|111000100⟩+β|011100100⟩)

= α|111100100⟩+β|011100100⟩

= $\alpha|744\rangle_o + \beta|344\rangle_o$

> $Co12 := Multiply(M1, Co11);$
> $Vr(Co12, 3);$ `$|V12\rangle$` $:= Multiply(T(Co12), St)[1, 1];$

$$Co12 := \begin{bmatrix} \textit{512 x 1 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix}$$

$$\beta_{|228\rangle, |344\rangle, |011100100\rangle}$$

$$\alpha_{|484\rangle, |744\rangle, |111100100\rangle}$$

$$|V12\rangle := \alpha\,|111100100\rangle + \beta\,|011100100\rangle \quad \textbf{(26)}$$

if qp = 1

V13 = M6 · V12

= M6(α|111100100⟩+β|011100100⟩

= α|011100100⟩+β|111100100⟩

= $\alpha|344\rangle_o + \beta|744\rangle_o$

which can be re-written as

V13 = (α|0⟩+β|1⟩) ⊗|11100100⟩

if qp = 2 then V13 = α|010100100⟩+β|110100100⟩ = $\alpha|244\rangle_o + \beta|644\rangle_o$

if qp = 3 then V13 = α|001100100⟩+β|101100100⟩ = $\alpha|144\rangle_o + \beta|544\rangle_o$

> $Co13 := Multiply(M6, Co12);$
$Vr(Co12, 3);$
$`|V13\rangle` := Multiply(T(Co13), St)[1, 1];$

$$Co13 := \begin{bmatrix} \text{512 x 1 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix}$$

$$\beta_{|228\rangle, |344\rangle, |011100100\rangle}$$

$$\alpha_{|484\rangle, |744\rangle, |111100100\rangle}$$

$$|V13\rangle := \alpha\, |011100100\rangle + \beta\, |111100100\rangle \tag{27}$$

The 9-qubit code:

$|q_1\, q_2\, q_3\, q_4\, q_5\, q_6\, q_7\, q_8\, q_9\rangle$:

qubit-flips: q_2q_3 = 11 1st qubit q_5q_6 = 11 4th qubit q_8q_9 = 11 7th qubit
= 10 2nd = 10 5th = 10 8th
= 01 3rd = 01 6th = 01 9th

phase-flip codes: q_4q_7 = 11 at 1st qubit
= 10 at 4th
= 01 at 7th

Examples:

$|q1\rangle \otimes |\, 11\ 00\ 00\ 00\,\rangle$: no phase-flips, qubit-flip at 1
$|q1\rangle \otimes |\, 10\ 00\ 00\ 00\,\rangle$: no phase-flips, qubit-flip at 2
$|q1\rangle \otimes |\, 01\ 00\ 00\ 00\,\rangle$: no phase-flips, qubit-flip at 3
$|q1\rangle \otimes |\, 00\ 10\ 01\ 00\,\rangle$: phase-flip at 1, no qubit-flips
$|q1\rangle \otimes |\, 00\ 10\ 00\ 00\,\rangle$: phase-flip at 4 no qubit-flips
$|q1\rangle \otimes |\, 00\ 00\ 01\ 00\,\rangle$: phase-flip at 7 no qubit-flips
$|q1\rangle \otimes |\, 11\ 10\ 01\ 00\,\rangle$: phase-flip & qubit flip at 1
$|q1\rangle \otimes |\, 00\ 10\ 10\ 00\,\rangle$: phase-flip & qubit-flip at 6