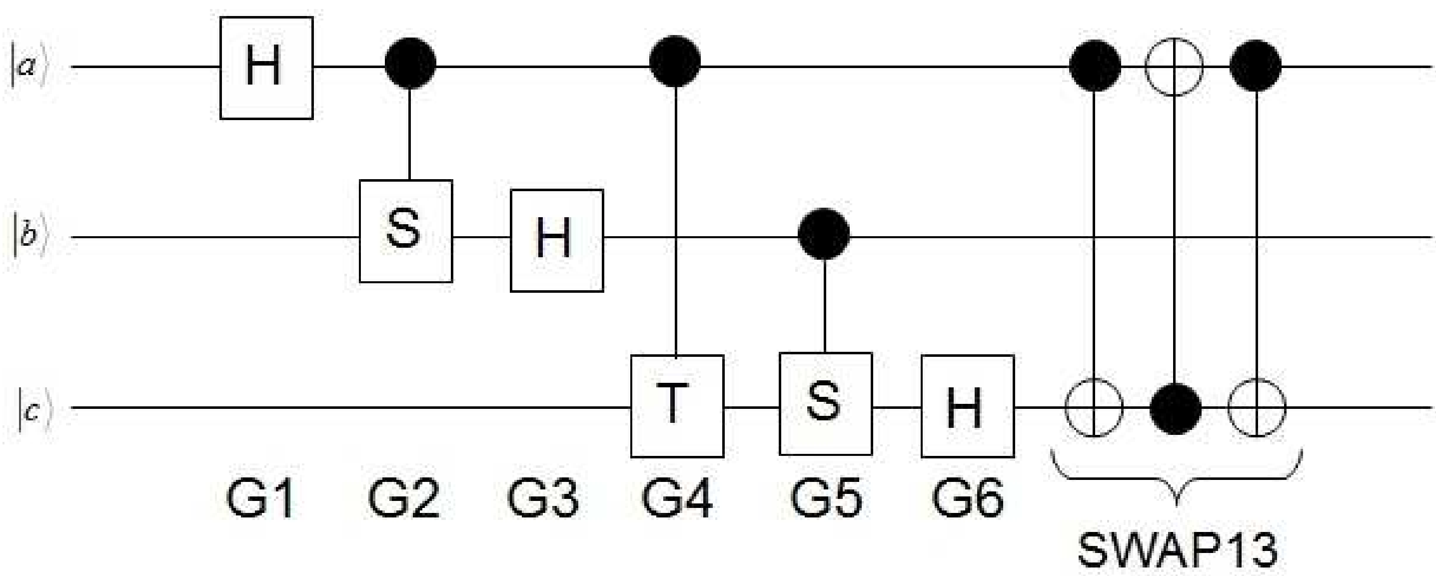
```
> restart;
> interface(warnlevel = 0) :          #   Maple 12
> with(LinearAlgebra) :
> with(Bits) :
```

Chapter 8 Problem 6




```
> TP :=proc(M1, M2) return KroneckerProduct(M1, M2) end proc:
> T := proc(x) return Transpose(x) end proc:
> VSte := proc(n)                    # Generates a list of  computational states for  n qubits
            local i, L;              #  e.g.  n= 2  ⇒ [ |00⟩  |01⟩  |10⟩  |11⟩ ]
            L := Matrix(1, 2^n);
            Settings(defaultbits = n);
            for i from 1 to 2^n do
               L[1, i] := cat(`|`, String(i − 1, msbfirst), "⟩");
            end do;
      #   print(L);
          return L;       # returns Matrix L
        end proc:
> Ceff :=proc(n)
             local i, L;              #  Generates a list of  computational state coefficients for  n qubits
             L := Matrix(1, 2^n);               #  e.g.  n= 2 ⇒ [c0 c1 c2 c3]
             for i from 1 to 2^n do
                L[1, i] := c[i − 1];         # c[i-1] represents the coefficient c_i
             end do;
       #   print(L);
          return L;        # returns Matrix L
          end proc:
```

Defining matrices

```
> I2 := IdentityMatrix(2) :
I8 := IdentityMatrix(8);
X := Matrix([[0, 1], [1, 0]]);
Z := Matrix([[1, 0], [0, −1]]);
S := Matrix([[1, 0], [0, I]]);
T := Matrix([[1, 0], [0, (1 + I)/'sqrt(2)']]);
H := 1/sqrt(2) (X + Z);
G23 := RowOperation(IdentityMatrix(4), [2, 3]);
```

$$I8 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S := \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{I} \end{bmatrix}$$

$$T := \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+\mathrm{I}}{\sqrt{2}} \end{bmatrix}$$

$$H := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$G23 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

Cntrl S gate; applies a phase shift $e^{\frac{i\pi}{2}}$ when control & target qubits are '1'

> **Us** := *Matrix*([[1, 0, 0, 0], [0, 1 , 0, 0], [0, 0, 1, 0], [0, 0, 0, *I*]]);

$$Us := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{I} \end{bmatrix} \tag{2}$$

Cntrl T gate; applies a phase shift $e^{\frac{i\pi}{4}}$ when control & target qubits are '1'

> **Ut** := $Matrix\left(\left[[1, 0, 0, 0], [0, 1 , 0, 0], [0, 0, 1, 0], \left[0, 0, 0, \frac{(1+I)}{'\sqrt{2}'}\right]\right]\right)$;

$$Ut := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1+\mathrm{I}}{\sqrt{2}} \end{bmatrix} \tag{3}$$

Three qubit 'SWAP' Gates:

> **SWAP** := *TP*(*G23*, *I2*);   # swaps the 1st and 2nd qubit

$$SWAP := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

> **SWAP13** := *RowOperation*(*RowOperation*(**I8**, [2, 5]), [4, 7]);   # swaps the 1st and last qubit

$$SWAP13 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

****

Shows the three-qubit 'SWAP' gates.

> $V := Ceff(3)$:
> $St := Transpose(VSte(3))$:

> $|\psi\rangle := Multiply(V, St)[1, 1]$;
> $|\psi'\rangle := Multiply(V, Multiply(\mathbf{SWAP}, St))[1, 1]$;

$$|\psi\rangle := c_0\,|000\rangle + c_1\,|001\rangle + c_2\,|010\rangle + c_3\,|011\rangle + c_4\,|100\rangle + c_5\,|101\rangle + c_6\,|110\rangle + c_7\,|111\rangle$$
$$|\psi'\rangle := c_0\,|000\rangle + c_1\,|001\rangle + c_2\,|100\rangle + c_3\,|101\rangle + c_4\,|010\rangle + c_5\,|011\rangle + c_6\,|110\rangle + c_7\,|111\rangle \tag{6}$$

> \`$|\psi\rangle$\` $:= Multiply(V, St)[1, 1]$;
> \`$|\psi'\rangle$\` $:= Multiply(V, Multiply(\mathbf{SWAP13}, St))[1, 1]$;

$$|\psi\rangle := c_0\,|000\rangle + c_1\,|001\rangle + c_2\,|010\rangle + c_3\,|011\rangle + c_4\,|100\rangle + c_5\,|101\rangle + c_6\,|110\rangle + c_7\,|111\rangle$$
$$|\psi'\rangle := c_0\,|000\rangle + c_1\,|100\rangle + c_2\,|010\rangle + c_3\,|110\rangle + c_4\,|001\rangle + c_5\,|101\rangle + c_6\,|011\rangle + c_7\,|111\rangle \tag{7}$$

****

Gate Configuration. There are seven stages: the first six are G1 - G6. The last stage is the SWAP13 gate

```
> G1 := TP(H, TP(I2, I2));
  G2 := TP(Us, I2);
  G3 := TP(I2, TP(H, I2));
  G4 := Multiply(SWAP,Multiply(TP(I2, Ut), SWAP));
  G5 := TP(I2, Us);
  G6 :=  TP(TP(I2, I2), H);
```

$$G1 := \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$G2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

$$G3 := \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$G4 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\mathrm{I}}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+\mathrm{I}}{\sqrt{2}} \end{bmatrix}$$

$$G5 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} \end{bmatrix}$$

$$G6 := \left[\begin{array}{cccccccc}
\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2}
\end{array}\right] \quad \textbf{(8)}$$

The composition of G1 - G6

> $G := Multiply(G6, Multiply(G5, Multiply(G4, Multiply(G3, Multiply(G2, G1)))));$

$$G := \left[\begin{array}{cccccccc}
\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} \\
\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} \\
\frac{1}{4}\sqrt{2} & \frac{1}{4}I\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}I\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}I\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}I\sqrt{2} \\
\frac{1}{4}\sqrt{2} & -\frac{1}{4}I\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}I\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}I\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}I\sqrt{2} \\
\frac{1}{4}\sqrt{2} & \frac{1}{4}+\frac{1}{4}I & \frac{1}{4}I\sqrt{2} & -\frac{1}{4}+\frac{1}{4}I & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}-\frac{1}{4}I & -\frac{1}{4}I\sqrt{2} & \frac{1}{4}-\frac{1}{4}I \\
\frac{1}{4}\sqrt{2} & -\frac{1}{4}-\frac{1}{4}I & \frac{1}{4}I\sqrt{2} & \frac{1}{4}-\frac{1}{4}I & -\frac{1}{4}\sqrt{2} & \frac{1}{4}+\frac{1}{4}I & -\frac{1}{4}I\sqrt{2} & -\frac{1}{4}+\frac{1}{4}I \\
\frac{1}{4}\sqrt{2} & -\frac{1}{4}+\frac{1}{4}I & -\frac{1}{4}I\sqrt{2} & \frac{1}{4}+\frac{1}{4}I & -\frac{1}{4}\sqrt{2} & \frac{1}{4}-\frac{1}{4}I & \frac{1}{4}I\sqrt{2} & -\frac{1}{4}-\frac{1}{4}I \\
\frac{1}{4}\sqrt{2} & \frac{1}{4}-\frac{1}{4}I & -\frac{1}{4}I\sqrt{2} & -\frac{1}{4}-\frac{1}{4}I & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}+\frac{1}{4}I & \frac{1}{4}I\sqrt{2} & \frac{1}{4}+\frac{1}{4}I
\end{array}\right] \quad \textbf{(9)}$$

Last Stage: the SWAP13 gate

> $U := Multiply(SWAP13, G);$

$$U := \begin{bmatrix}
\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} \\
\frac{1}{4}\sqrt{2} & \frac{1}{4}+\frac{1}{4}\,\mathrm{I} & \frac{1}{4}\,\mathrm{I}\sqrt{2} & -\frac{1}{4}+\frac{1}{4}\,\mathrm{I} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}-\frac{1}{4}\,\mathrm{I} & -\frac{1}{4}\,\mathrm{I}\sqrt{2} & \frac{1}{4}-\frac{1}{4}\,\mathrm{I} \\
\frac{1}{4}\sqrt{2} & \frac{1}{4}\,\mathrm{I}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\,\mathrm{I}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\,\mathrm{I}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\,\mathrm{I}\sqrt{2} \\
\frac{1}{4}\sqrt{2} & -\frac{1}{4}+\frac{1}{4}\,\mathrm{I} & -\frac{1}{4}\,\mathrm{I}\sqrt{2} & \frac{1}{4}+\frac{1}{4}\,\mathrm{I} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}-\frac{1}{4}\,\mathrm{I} & \frac{1}{4}\,\mathrm{I}\sqrt{2} & -\frac{1}{4}-\frac{1}{4}\,\mathrm{I} \\
\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} \\
\frac{1}{4}\sqrt{2} & -\frac{1}{4}-\frac{1}{4}\,\mathrm{I} & \frac{1}{4}\,\mathrm{I}\sqrt{2} & \frac{1}{4}-\frac{1}{4}\,\mathrm{I} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}+\frac{1}{4}\,\mathrm{I} & -\frac{1}{4}\,\mathrm{I}\sqrt{2} & -\frac{1}{4}+\frac{1}{4}\,\mathrm{I} \\
\frac{1}{4}\sqrt{2} & -\frac{1}{4}\,\mathrm{I}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\,\mathrm{I}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\,\mathrm{I}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\,\mathrm{I}\sqrt{2} \\
\frac{1}{4}\sqrt{2} & \frac{1}{4}-\frac{1}{4}\,\mathrm{I} & -\frac{1}{4}\,\mathrm{I}\sqrt{2} & -\frac{1}{4}-\frac{1}{4}\,\mathrm{I} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}+\frac{1}{4}\,\mathrm{I} & \frac{1}{4}\,\mathrm{I}\sqrt{2} & \frac{1}{4}+\frac{1}{4}\,\mathrm{I}
\end{bmatrix} \tag{10}$$

Multiplying by $\sqrt{8}$

> $U := \sqrt{8}\,U:$
> $'\sqrt{8}\cdot U' = U;$

$$\sqrt{8}\;U \tag{11}$$
$$= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \left(\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & \mathrm{I} & \left(-\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & -1 & \left(-\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & -\mathrm{I} & \left(\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} \\
1 & \mathrm{I} & -1 & -\mathrm{I} & 1 & \mathrm{I} & -1 & -\mathrm{I} \\
1 & \left(-\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & -\mathrm{I} & \left(\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & -1 & \left(\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & \mathrm{I} & \left(-\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & \left(-\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & \mathrm{I} & \left(\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & -1 & \left(\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & -\mathrm{I} & \left(-\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} \\
1 & -\mathrm{I} & -1 & \mathrm{I} & 1 & -\mathrm{I} & -1 & \mathrm{I} \\
1 & \left(\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & -\mathrm{I} & \left(-\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & -1 & \left(-\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\right)\sqrt{2} & \mathrm{I} & \left(\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\right)\sqrt{2}
\end{bmatrix}$$

Making the following substitutions:

> $'\sqrt{8}\cdot U' = subs\left(\frac{1+I}{\sqrt{2}} = \alpha, \frac{-1-I}{\sqrt{2}} = -\alpha, \frac{1-I}{\sqrt{2}} = \beta, \frac{-1+I}{\sqrt{2}} = -\beta, U\right);$

$$\sqrt{8}\;U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & I & -\beta & -1 & -\alpha & -I & \beta \\ 1 & I & -1 & -I & 1 & I & -1 & -I \\ 1 & -\beta & -I & \alpha & -1 & \beta & I & -\alpha \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\alpha & I & \beta & -1 & \alpha & -I & -\beta \\ 1 & -I & -1 & I & 1 & -I & -1 & I \\ 1 & \beta & -I & -\alpha & -1 & -\beta & I & \alpha \end{bmatrix} \tag{12}$$