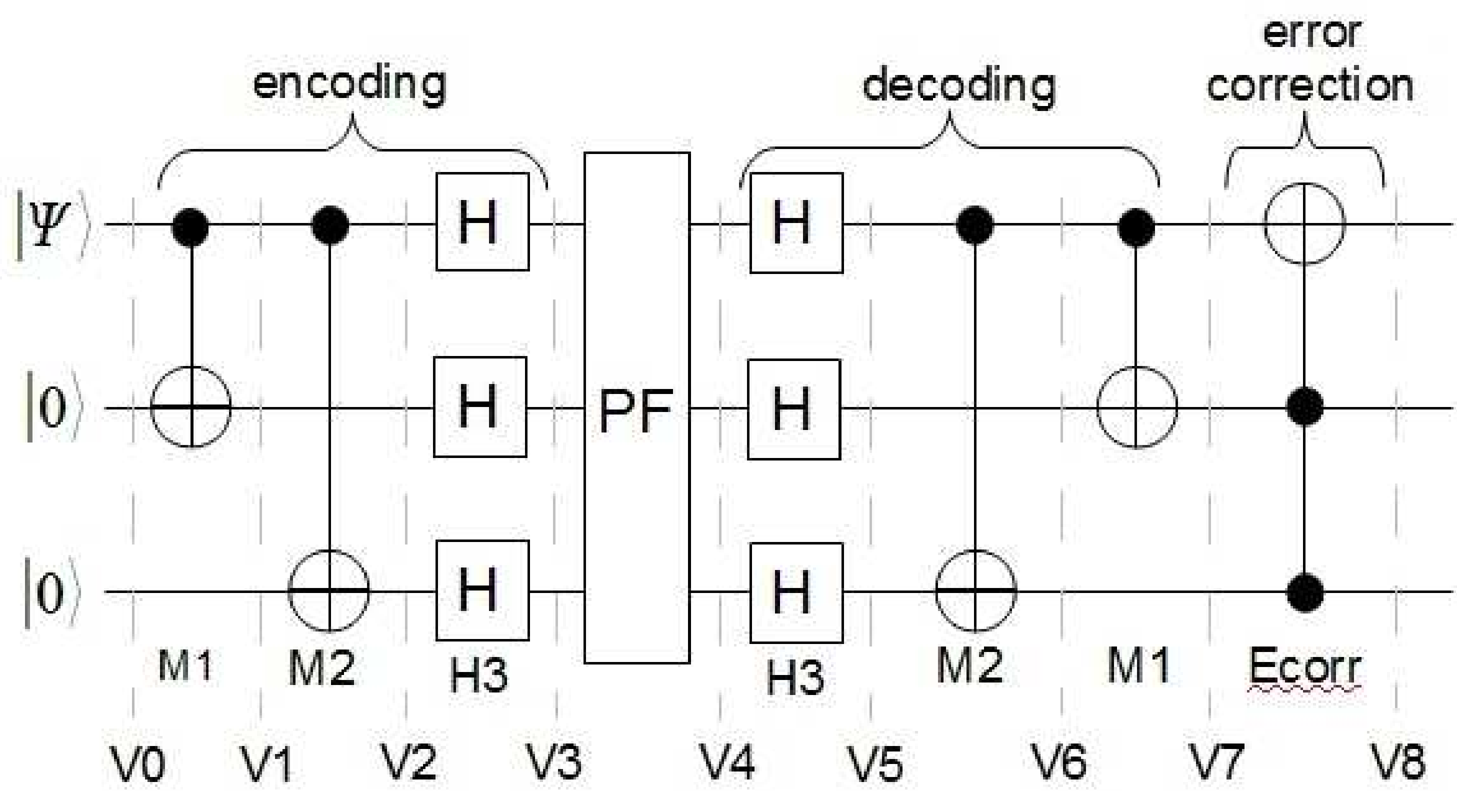
```
> restart :
> interface(warnlevel = 0) :            #  Maple 12
> with(LinearAlgebra) :
> with(Bits) :
```



Utility functions

```
> K := proc(a, b)  return KroneckerProduct(a, b)  end proc:
> T :=proc(x) return Transpose(x) end proc:
> VSte := proc(n)                    # Generates a list of  computational states for  n qubits
           local i, L;               #  e.g.  n= 2  ⇒ [ |00⟩ |01⟩ |10⟩ |11⟩ ]
           L := Matrix(1, 2^n);
           Settings(defaultbits = n);
           for i from 1 to 2^n do
               L[1, i] := cat(`|`, String(i − 1, msbfirst), "⟩");
           end do;
        #    print(L);
          return L;       # returns Matrix L
        end proc:
```

Utility matrices

```
> I2 := IdentityMatrix(2);
```

$$I2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1}$$

```
> I4 := IdentityMatrix(4);
```

$$I4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

```
> I8 := IdentityMatrix(8);
```

$$I8 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

Utility Operators

```
> Uz := RowOperation(I2, 2, -1);       # phase-flip operator
```

$$Uz := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (4)$$

```
> CNOT := RowOperation(I4, [3, 4]);
```

$$CNOT := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (5)$$

```
> G23 := RowOperation(I4, [2, 3]);     # qubit-exchange operator
```

$$G23 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

$$> H := \frac{1}{\sqrt{2}} Matrix([[1, 1], [1, -1]]);$$

$$H := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix} \quad (7)$$

M1, M2, H3, PF, and Ecorr matrices/operators

> M1 := $K(CNOT, I2)$;

$$M1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{8}$$

> M2 := $Multiply(K(G23, I2), Multiply(K(I2, CNOT), K(G23, I2)))$;

$$M2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{9}$$

> $H3 := K(H, K(H, H))$:
$'\sqrt{8} \cdot H3' = simplify(\sqrt{8}\, H3)$;

$$\sqrt{8}\ H3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \tag{10}$$

> $PF := K(Uz, I4);$ # *Matrix/operator for a phase-flip error on the first qubit*

$$PF := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad (11)$$

Ecorr matrix/operator is a variation of the Toffoli gate. The 2nd and 3rd qubits are the control qubits. When the state of both qubits equal "1" the 1st qubit is "flipped"

$$c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + \mathbf{c_3}|\mathbf{011}\rangle + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + \mathbf{c_7}|\mathbf{111}\rangle$$

$$c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + \mathbf{c_3}|\mathbf{111}\rangle + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + \mathbf{c_7}|\mathbf{011}\rangle$$

> $Ecorr := RowOperation(I8, [8, 4]);$ # *Error Correction matrix*

$$Ecorr := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (12)$$

$$\mathbf{|V0\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \otimes |0\rangle = \alpha|000\rangle + \beta|100\rangle}$$

```
> n := 3 :
  q1 := Matrix([[α], [β]]) :    #  α|0⟩ + β|1⟩
  q2 := Matrix([[1], [0]]) :    #  |0⟩
  q3 := Matrix([[1], [0]]) :    #  |0⟩
  Co0 := K(K(q1, q2), q3) :
  St := Transpose(VSte(n)) :
  |V0⟩ := Multiply(T(Co0), St)[1, 1];
```

$$|V0\rangle := \alpha\ |000\rangle + \beta\ |100\rangle \quad (13)$$

$$
\begin{aligned}
|\mathbf{V1}\rangle &= \mathbf{M1}\cdot|\mathbf{V0}\rangle \\
&= \mathbf{M1}(\alpha|\mathbf{000}\rangle + \beta|\mathbf{100}\rangle) \\
&= \alpha|\mathbf{000}\rangle + \beta|\mathbf{110}\rangle
\end{aligned}
$$

> $Co1 := Multiply(M1, Co0):$
> $|V1\rangle := factor(Multiply(T(Co1), St)[1, 1]);$

$$|V1\rangle := \alpha\ |000\rangle + \beta\ |110\rangle \tag{14}$$

$$
\begin{aligned}
|\mathbf{V2}\rangle &= \mathbf{M2}\cdot|\mathbf{V1}\rangle \\
&= \mathbf{M2}(\alpha|\mathbf{000}\rangle + \beta|\mathbf{110}\rangle) \\
&= \alpha|\mathbf{000}\rangle + \beta|\mathbf{111}\rangle
\end{aligned}
$$

> $\mathrm{Co2} := Multiply(M2, Co1):$
> $|V2\rangle := factor(Multiply(T(Co2), St)[1, 1]);$

$$|V2\rangle := \alpha\ |000\rangle + \beta\ |111\rangle \tag{15}$$

The Hadamard H3 matrix/operator

> $Co3 := simplify\left(\sqrt{8}\cdot Multiply(H3, Co2)\right):$
> $|V3\rangle := Multiply(T(Co3), St)[1, 1];$

$$|V3\rangle := (\alpha+\beta)\ |000\rangle + (\alpha-\beta)\ |001\rangle + (\alpha-\beta)\ |010\rangle + (\alpha+\beta)\ |011\rangle + (\alpha-\beta)\ |100\rangle + (\alpha+\beta)\ |101\rangle + (\alpha+\beta)\ |110\rangle + (\alpha-\beta)\ |111\rangle \tag{16}$$

The encoding operator/matrix

> $Encode := Multiply(H3, Multiply(M2, M1)):$
> $'\sqrt{8}\cdot Encode' = \sqrt{8}\,Encode;$
> $\mathrm{Co3} := \mathrm{simplify}\left(\sqrt{8}\cdot \mathrm{Multiply}(Encode, Co0)\right):$
> $|\Psi 3\rangle := Multiply(T(Co3), St)[1, 1];$

$$\sqrt{8}\ Encode = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

$$|\Psi 3\rangle := (\alpha+\beta)\ |000\rangle + (\alpha-\beta)\ |001\rangle + (\alpha-\beta)\ |010\rangle + (\alpha+\beta)\ |011\rangle + (\alpha-\beta)\ |100\rangle + (\alpha+\beta)\ |101\rangle + (\alpha+\beta)\ |110\rangle + (\alpha-\beta)\ |111\rangle \tag{17}$$

A phase-flip error on the first qubit

$$\mathbf{V4} = \alpha|-++\rangle + \beta|+--\rangle$$

> $Co4 := simplify(Multiply(PF, Co3))$:
> $|V4\rangle := Multiply(T(Co4), St)[1, 1]$;

$$|V4\rangle := (\alpha+\beta)\ |000\rangle + (\alpha-\beta)\ |001\rangle + (\alpha-\beta)\ |010\rangle + (\alpha+\beta)\ |011\rangle + (-\alpha+\beta)\ |100\rangle + (-\alpha-\beta)\ |101\rangle + (-\alpha-\beta)\ |110\rangle + (-\alpha+\beta)\ |111\rangle \quad \textbf{(18)}$$

The Hadamard H3 matrix/operator

$$\mathbf{V5} = \alpha|\mathbf{100}\rangle + \beta|\mathbf{011}\rangle$$

> $Co5 := simplify\left(\frac{1}{\sqrt{8}} \cdot Multiply(H3, Co4)\right)$:
> $|V5\rangle := Multiply(T(Co5), St)[1, 1]$;

$$|V5\rangle := \beta\ |011\rangle + \alpha\ |100\rangle \quad \textbf{(19)}$$

$$\mathbf{V6} = \mathbf{M2} \cdot \mathbf{V5}$$
$$= \mathbf{M2}(\alpha|\mathbf{100}\rangle + \beta|\mathbf{011}\rangle)$$
$$= \alpha|\mathbf{101}\rangle + \beta|\mathbf{011}\rangle$$

> $\mathrm{Co6} := Multiply(M2, Co5)$:
> $|V6\rangle := Multiply(T(Co6), St)[1, 1]$;

$$|V6\rangle := \beta\ |011\rangle + \alpha\ |101\rangle \quad \textbf{(20)}$$

$$\mathbf{V7} = \mathbf{M1} \cdot \mathbf{V6}$$
$$= \mathbf{M1}(\alpha|\mathbf{101}\rangle + \beta|\mathbf{011}\rangle)$$
$$= \alpha|\mathbf{111}\rangle + \beta|\mathbf{011}\rangle$$

> $Co7 := Multiply(M1, Co6)$:
> $|V7\rangle := (Multiply(T(Co7), St)[1, 1])$;

$$|V7\rangle := \beta\ |011\rangle + \alpha\ |111\rangle \quad \textbf{(21)}$$

The decoding operator/matrix

> $Decode := simplify(Multiply(M1, Multiply(M2, H3)))$:
$'\sqrt{8} \cdot Decode' = \sqrt{8}\, Decode;$
$Co7 := simplify\left(\frac{1}{\sqrt{8}} \cdot Multiply(Decode, Co4)\right)$:
$|V7\rangle := Multiply(T(Co7), St)[1, 1];$

$$\sqrt{8}\ Decode = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

$$|V7\rangle := \beta\ |011\rangle + \alpha\ |111\rangle \tag{22}$$

Encode x Decode = I_8

> $Multiply(Encode, Decode);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

$$\mathbf{V8 = Ecorr \cdot V7}$$
$$\mathbf{= Ecorr(\alpha|111\rangle + \beta|011\rangle)}$$
$$\mathbf{= \alpha|011\rangle + \beta|111\rangle}$$
$$\mathbf{= (\alpha|0\rangle + \beta|1\rangle) \otimes |1\rangle \otimes |1\rangle}$$

> $\mathrm{Co8} := \mathrm{simplify}(\mathrm{Multiply}(Ecorr, Co7))$:
$|V8\rangle := Multiply(T(Co8), St)[1, 1];$

$$|V8\rangle := \alpha\ |011\rangle + \beta\ |111\rangle \tag{24}$$

Compare to V0

$$\mathbf{|V0\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \otimes |0\rangle = \alpha|000\rangle + \beta|100\rangle}$$

> $|V0\rangle := Multiply(T(Co0), St)[1, 1];$

$$|V0\rangle := \alpha\ |000\rangle + \beta\ |100\rangle \tag{25}$$