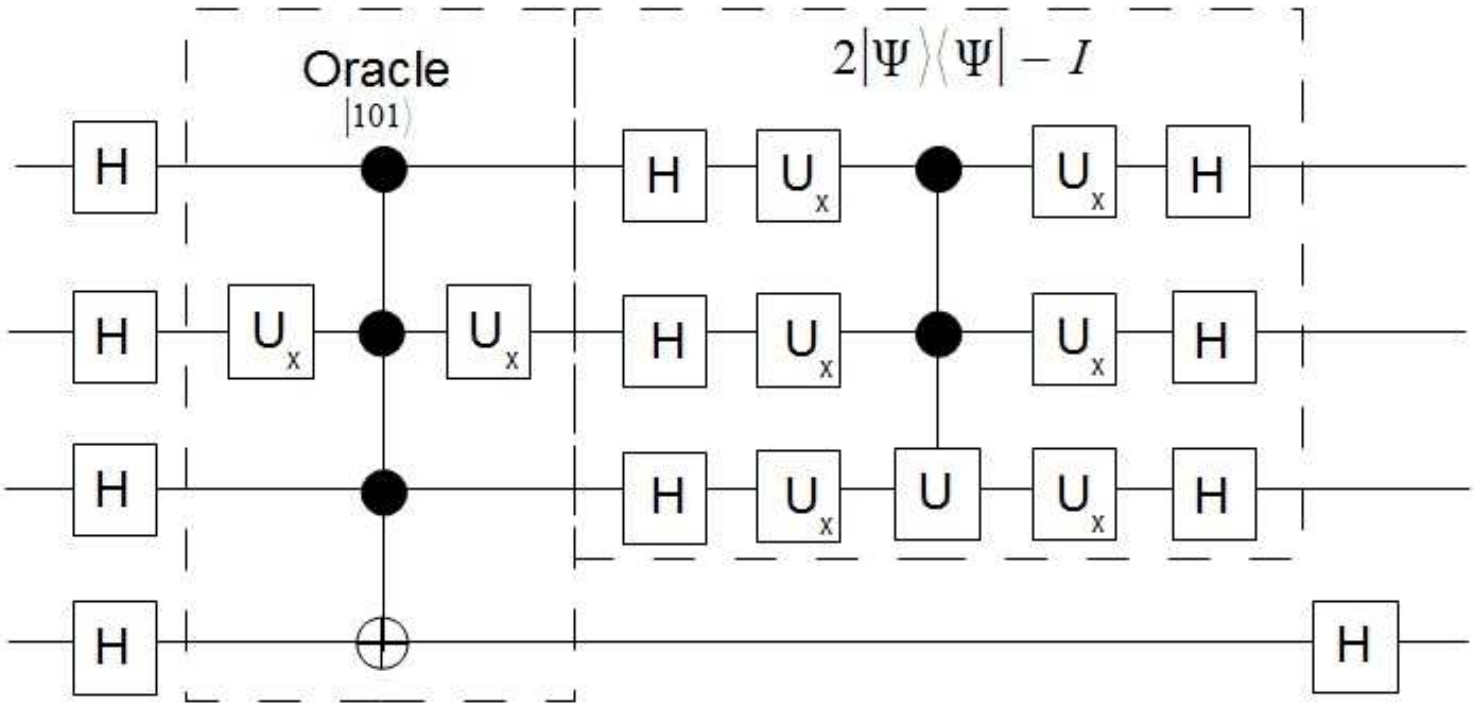
```
> restart;
> interface(warnlevel = 0) :   # Maple 12
> interface(rtablesize = 32) :
> with(LinearAlgebra) :
> with(Bits) :
```

Grover's Algorithm



1 out of 8 search. Need 3 qubits to generate the computational basis

```
> n := 3 :
```

Goal is the state of interest. For example, 5 ⇒ |101⟩

```
> g := 5 :
```

Functions

```
> TP := proc(M1, M2)
        KroneckerProduct(M1, M2);
      end proc:
```

```
> VSte := proc(n)                    # Generates a list of computational states for n qubits
            local i, L;              # e.g. n= 2 ⇒ [ |00⟩ |01⟩ |10⟩ |11⟩ ]
            L := Matrix(1, 2^n);
            Settings(defaultbits = n);
            for i from 1 to 2^n do
                L[1, i] := cat(`|`, String(i - 1, msbfirst), "⟩");
            end do;
        #   print(L);
            return L;        # returns Matrix L
          end proc:
```

```
> NumCoe := proc(n, s)                    # generates the initial coefficients
            local p, m, i, `1`, `0`;
            `0` := Matrix([[1], [0]]) :
            `1` := Matrix([[0], [1]]) :
            p := Multiply(H, `0`) :
            m := Multiply(H, `1`) :
            if s = 0 then t := p else t := TP(p, m) end if;
            for i from 1 to n - 1 do
                t := TP(p, t) :
            end do;
            return t;
          end proc:
```

```
> MaxCo := proc(L, n)                    # returns the location of largest coefficient
           local x, y, i, N, loc;
           x := 0 :
           N := 2^(n + 1);
           for i from 1 to N do
             y := abs(evalf( L[i, 1]));
             if x < y then  loc := i : x := y end if
           end do;
           return loc :
         end proc:
```

Operators/Matrices

```
> I2 := IdentityMatrix(2) :          # Identity Matrices
  I4 := IdentityMatrix(4) :
  I8 := IdentityMatrix(8) :
  I16 := IdentityMatrix(16) :
```

```
> Ux := RowOperation(I2, [1, 2]);
  H := 1/sqrt(2) Matrix([[1, 1], [1, -1]]);    # Hadamard
  Φ := e^(I·π) I2;  # phase
  # Φ := I2;
  Gt := RowOperation(I8, [7, 8]);        # Toffoli gate
  Gt3 := RowOperation(I16, [15, 16]);  # Oracle gate
  CCU := (Multiply(TP(Φ, TP(I2, H)), Multiply(Gt, TP(I4, H))));
  H3 := TP(H, TP(H, H));
  Ux3 := TP(Ux, TP(Ux, Ux));
```

$$Ux := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$H := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$\Phi := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Gt := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$Gt3 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$CCU := \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H3 := \begin{bmatrix} \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} \end{bmatrix}$$

$$Ux3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(1)

The Oracles

```
> if g = 0 then
      Oracle := Multiply(TP(Ux, TP(Ux, TP(Ux, I2))), Multiply(Gt3, (TP(Ux, TP(Ux, TP(Ux, I2)))))));
    # |000⟩
  end if;
  if g = 1 then
      Oracle := Multiply(TP(Ux, TP(Ux, I4)), Multiply(Gt3, TP(Ux, TP(Ux, I4))));  # goal is: |001⟩
  end if;
  if g = 2 then
      Oracle := Multiply(TP(Ux, TP(I2, TP(Ux, I2))), Multiply(Gt3, (TP(Ux, TP(I2, TP(Ux, I2)))))));
    # |010⟩
  end if;
  if g = 3 then
      Oracle := Multiply(TP(Ux, TP(I2, I4)), Multiply(Gt3, TP(Ux, TP(I2, I4))));  # goal is: |011⟩
  end if;
  if g = 4 then
      Oracle := Multiply(TP(I2, TP(Ux, TP(Ux, I2))), Multiply(Gt3, (TP(I2, TP(Ux, TP(Ux, I2)))))));
    # |100⟩
  end if;
  if g = 5 then     # `Oracle shown above`
      Oracle := Multiply(TP(I2, TP(Ux, I4)), Multiply(Gt3, TP(I2, TP(Ux, I4))));  # goal is: |101⟩
  end if;
  if g = 6 then
      Oracle := Multiply(TP(I4, TP(Ux, I2)), Multiply(Gt3, TP(I4, TP(Ux, I2))));  # goal is: |110⟩
  end if;
  if g = 7 then
      Oracle := Gt3 ;  # goal is: |111⟩
  end if;
```

$$Oracle := \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \quad \textbf{(2)}$$

The - (I - 2|Ψ⟩⟨Ψ|) Operator

```
#   N := 2^n :
#   ρ := ConstantMatrix(1, N) :
#   ℙ := (2/N)·ρ − IdentityMatrix(N);  # The 2|Ψ⟩⟨Ψ| - I Operator
#   𝕄 := TP(ℙ, I2);

ℙ := Multiply(H3, Multiply(Ux3, Multiply(CCU, Multiply(Ux3, H3))));
𝕄 := TP(ℙ, I2) :
```

$$\mathbb{P} := \begin{bmatrix} -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \end{bmatrix} \quad \textbf{(3)}$$

Computational basis

```
St := VSte(n) :
Co0 := NumCoe(n, 0) :
|Ψ0⟩ := factor(Multiply(St, Co0)[1, 1]);
```

$$|\Psi 0\rangle := \frac{1}{4}\sqrt{2}\,(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \quad \textbf{(4)}$$

Computational basis with the "Oracle qubit"

```
St := VSte(n + 1) :
Co1 := NumCoe(n, 1) :
|Ψ1⟩ := factor(Multiply(St, Co1)[1, 1]);
```

$$|\Psi 1\rangle := \frac{1}{4}|0000\rangle - \frac{1}{4}|0001\rangle + \frac{1}{4}|0010\rangle - \frac{1}{4}|0011\rangle + \frac{1}{4}|0100\rangle - \frac{1}{4}|0101\rangle + \frac{1}{4}|0110\rangle - \frac{1}{4}|0111\rangle + \frac{1}{4}|1000\rangle - \frac{1}{4}|1001\rangle + \frac{1}{4}|1010\rangle - \frac{1}{4}|1011\rangle + \frac{1}{4}|1100\rangle - \frac{1}{4}|1101\rangle + \frac{1}{4}|1110\rangle - \frac{1}{4}|1111\rangle \quad (5)$$

First Oracle query

```
Co2 := Multiply(Oracle, Co1) :
|Ψ2⟩ := factor( Multiply(St, Co2)[1, 1]);
```

$$|\Psi 2\rangle := \frac{1}{4}|0000\rangle - \frac{1}{4}|0001\rangle + \frac{1}{4}|0010\rangle - \frac{1}{4}|0011\rangle + \frac{1}{4}|0100\rangle - \frac{1}{4}|0101\rangle + \frac{1}{4}|0110\rangle - \frac{1}{4}|0111\rangle + \frac{1}{4}|1000\rangle - \frac{1}{4}|1001\rangle - \frac{1}{4}|1010\rangle + \frac{1}{4}|1011\rangle + \frac{1}{4}|1100\rangle - \frac{1}{4}|1101\rangle + \frac{1}{4}|1110\rangle - \frac{1}{4}|1111\rangle \quad (6)$$

```
Co3 := Multiply(M, Co2) :
|Ψ3⟩ := factor( Multiply(St, Co3)[1, 1]);
```

$$|\Psi 3\rangle := \frac{1}{8}|0000\rangle - \frac{1}{8}|0001\rangle + \frac{1}{8}|0010\rangle - \frac{1}{8}|0011\rangle + \frac{1}{8}|0100\rangle - \frac{1}{8}|0101\rangle + \frac{1}{8}|0110\rangle - \frac{1}{8}|0111\rangle + \frac{1}{8}|1000\rangle - \frac{1}{8}|1001\rangle + \frac{5}{8}|1010\rangle - \frac{5}{8}|1011\rangle + \frac{1}{8}|1100\rangle - \frac{1}{8}|1101\rangle + \frac{1}{8}|1110\rangle - \frac{1}{8}|1111\rangle \quad (7)$$

Recover the Oracle qubit

```
Co4 := Multiply(TP(I2, TP(I2, TP(I2, H))), Co3) :
|Ψ4⟩ := factor( Multiply(St, Co4)[1, 1]);
```

$$|\Psi 4\rangle := \frac{1}{8}\sqrt{2}\,(|0001\rangle + |0011\rangle + |0101\rangle + |0111\rangle + |1001\rangle + 5|1011\rangle + |1101\rangle + |1111\rangle) \quad (8)$$

Preview result of the first pass

```
l := MaxCo(Co4, 3) :
State := St[1, l];
Probability := (evalf(Co4[l, 1])^2)·100;
```

$$State := |1011\rangle$$

$$Probability := 78.12499995 \quad (9)$$

Second query

```
> Co5 := Multiply(Oracle, Co3) :
  |Ψ5⟩ := factor( Multiply(St, Co5)[1, 1]);
```

$$|\Psi 5\rangle := \frac{1}{8}|0000\rangle - \frac{1}{8}|0001\rangle + \frac{1}{8}|0010\rangle - \frac{1}{8}|0011\rangle + \frac{1}{8}|0100\rangle - \frac{1}{8}|0101\rangle + \frac{1}{8}|0110\rangle - \frac{1}{8}|0111\rangle + \frac{1}{8}|1000\rangle - \frac{1}{8}|1001\rangle - \frac{5}{8}|1010\rangle + \frac{5}{8}|1011\rangle + \frac{1}{8}|1100\rangle - \frac{1}{8}|1101\rangle + \frac{1}{8}|1110\rangle - \frac{1}{8}|1111\rangle \quad \textbf{(10)}$$

```
> Co6 := Multiply(M, Co5) :
  |Ψ6⟩ := factor( Multiply(St, Co6)[1, 1]);
```

$$|\Psi 6\rangle := -\frac{1}{16}|0000\rangle + \frac{1}{16}|0001\rangle - \frac{1}{16}|0010\rangle + \frac{1}{16}|0011\rangle - \frac{1}{16}|0100\rangle + \frac{1}{16}|0101\rangle - \frac{1}{16}|0110\rangle + \frac{1}{16}|0111\rangle - \frac{1}{16}|1000\rangle + \frac{1}{16}|1001\rangle + \frac{11}{16}|1010\rangle - \frac{11}{16}|1011\rangle - \frac{1}{16}|1100\rangle + \frac{1}{16}|1101\rangle - \frac{1}{16}|1110\rangle + \frac{1}{16}|1111\rangle \quad \textbf{(11)}$$

Recover the Oracle qubit

```
> Co7 := Multiply(TP(I2, TP(I2, TP(I2, H))), Co6) :
  |Ψ7⟩ := factor( Multiply(St, Co7)[1, 1]);
```

$$|\Psi 7\rangle := -\frac{1}{16}\sqrt{2}\,(|0001\rangle + |0011\rangle + |0101\rangle + |0111\rangle + |1001\rangle - 11\,|1011\rangle + |1101\rangle + |1111\rangle) \quad \textbf{(12)}$$

Preview result of the second pass

```
> l := MaxCo(Co7, n) :
  State := St[1, l];
  Probability := (evalf(Co7[l, 1]))^2 · 100;
```

$$State := |1011\rangle$$

$$Probability := 94.53124996 \quad \textbf{(13)}$$