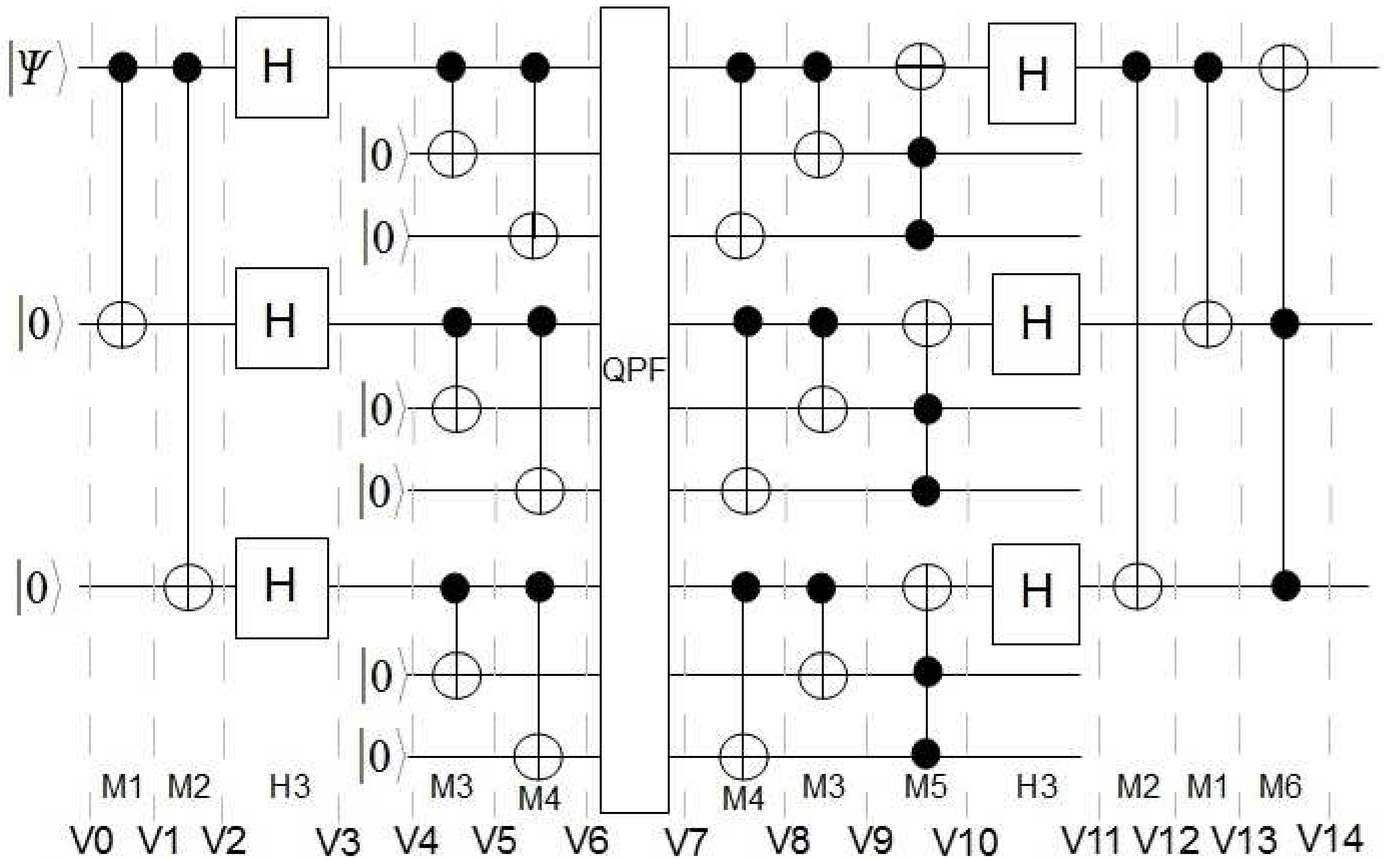
```
> restart :
> interface(warnlevel = 0) :      # Maple 12``
> with(LinearAlgebra) :
> with(Bits) :
> Settings(defaultbits = 9) :
```



Execution flag

qp = 1 : qubit-flip and phase-flip at 1st qubit

= 2 : qubit-flip and phase-flip at 2

= 3 : qubit-flip and phase-flip at 3

```
> qp := 1 :
```

Utility functions

```
> K := proc(a, b) return KroneckerProduct(a, b) end proc:
> T := proc(x) return Transpose(x) end proc:
```

```
> VSte := proc(n)                    # Generates a list of  computational states for  n qubits
          local i, L;               #  e.g.  n= 2  ⇒ [ |00⟩ |01⟩ |10⟩ |11⟩ ]
          L := Matrix(1, 2^n);
          Settings(defaultbits = n);
          for i from 1 to 2^n do
              L[1, i] := cat(`|`, String(i − 1, msbfirst), "⟩");
          end do;
      #    print(L);
        return L;       # returns Matrix L
      end proc:
> Vr := proc(x, y)           # prints out the 512 by 1 matrix as (±α±β) ( representation of the state)
           local i, j, b, d, o;
      # example: α|1001⟩ + β|1001⟩ →(α + β) |009⟩                Vr(X, 1) dec
           for i from 1 to 512 do       #                      → (α + β) |009⟩,|012⟩          Vr(X,
      2) dec, oct
               if x[i, 1] ≠ 0 then      #                      → (α + β) |009⟩,|012⟩,|1001⟩  Vr(X,
      2) dec, oct, bin
                  j := i − 1;
                  if  j ≤ 10 then
                     d := cat(`|00`, j, "⟩");
                  elif j ≤ 99 then
                     d := cat(`|0`, j, "⟩");
                  else
                     d := cat(`|`, j, "⟩");
                  end if;
                  if  j ≤ 7 then
                     o := cat(`|00`, convert( j, octal), "⟩");
                  elif j ≤ 63 then                          # j ≤ 77_o
                     o := cat(`|0`, convert( j, octal), "⟩");
                  else
                     o := cat(`|`, convert( j, octal), "⟩");
                  end if;
                  b := cat(`|`, String( j, msbfirst), "⟩");
                 if  y = 1 then print( (x[i, 1]) [d])
                     elif  y = 2 then  print( (x[i, 1]) [d, o])
                     elif  y = 3 then print( (x[i, 1]) [d, o, b]);
                  end if;
              end;
           end do;
          end proc:
```

Utility matrices

```
> I2 := IdentityMatrix(2) :
> I4 := IdentityMatrix(4) :
> I8 := IdentityMatrix(8) :
```

Utility Operators

> $Uz := RowOperation(I2, 2, -1);$ # *phase-flip operator*

$$Uz := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

> $Ux := RowOperation(I2, [1, 2]);$ # *qubit-flip operator*

$$Ux := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

> $CNOT := RowOperation(I4, [3, 4]);$

$$CNOT := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3}$$

> $G23 := RowOperation(I4, [2, 3]);$ # *qubit-exchange operator*

$$G23 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

> $H := \frac{1}{\sqrt{2}} Matrix([[1, 1], [1, -1]]);$

$$H := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix} \tag{5}$$

M1 - M6, H3, and QPF matrices/operators

> $\mathrm{M1} := K(CNOT, I2);$

$$M1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{6}$$

> M2 := $Multiply(\ K(G23, I2), Multiply(K(I2, CNOT),\ K(G23, I2)))$;

$$M2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{7}$$

> H3 := K(H, K(H, H)) :
'$\sqrt{8}\cdot$H3' = $\sqrt{8}$ H3;

$$\sqrt{8}\ H3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \tag{8}$$

> M3 := K(K(K(CNOT, I2), K(CNOT, I2)), K(CNOT, I2)); # *large matrix* 2^9 *by* 2^9

$$M3 := \begin{bmatrix} \textit{512 x 512 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \tag{9}$$

> M4 := K(Multiply(K(G23, I2), Multiply(K(I2, CNOT), K(G23, I2))),
K(Multiply(K(G23, I2), Multiply(K(I2, CNOT), K(G23, I2))),
Multiply(K(G23, I2), Multiply(K(I2, CNOT), K(G23, I2)))));

$$M4 := \begin{bmatrix} \textit{512 x 512 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \tag{10}$$

M6 matrix/operator is a variation of the Toffoli gate. The 2nd and 3rd qubits are the control qubits. When the state of both qubits equal "1" the 1st qubit is "flipped"

$$c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + \mathbf{c_3}|\mathbf{011}\rangle + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + \mathbf{c_7}|\mathbf{111}\rangle$$

$$c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + \mathbf{c_3}|\mathbf{111}\rangle + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + \mathbf{c_7}|\mathbf{011}\rangle$$

> M6 := RowOperation(I8, [8, 4]);

$$M6 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (11)$$

> M5 := K(K(M6, M6), M6);

$$M5 := \begin{bmatrix} \text{512 x 512 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix} \quad (12)$$

QPF: qubit error operator

```
> if qp = 1 then
    QPF := K(Multiply(Ux, Uz), IdentityMatrix(2^8));
      # phase-flip & qubit-flip errors on the first qubit
  elif qp = 2 then
      QPF := K(I2, K(Multiply(Ux, Uz), IdentityMatrix(2^7))); # phase-flip & qubit-flip at 2
  elif qp = 3 then
      QPF := K(I4, K(Multiply(Ux, Uz), IdentityMatrix(2^6))); # phase-flip  & qubit-flip at 3
  end if;
```

$$QPF := \begin{bmatrix} \text{512 x 512 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix} \quad (13)$$

$$\mathbf{V0} = (\alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle) \otimes |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle = \alpha|\mathbf{000}\rangle + \beta|\mathbf{100}\rangle$$

The decimal representation: $\alpha|\mathbf{000}\rangle + \beta|\mathbf{100}\rangle \Rightarrow \alpha|\mathbf{0}\rangle + \beta|\mathbf{4}\rangle$

The octal representation: $\alpha|\mathbf{000}\rangle + \beta|\mathbf{100}\rangle \Rightarrow \alpha\,|\mathbf{000}\rangle_{\mathbf{o}} + \beta|\mathbf{004}\rangle_{\mathbf{o}}$

```
> n := 3 :
  q1 := Matrix([[α], [β]]) :    #  α|0⟩ + β|1⟩
  q2 := Matrix([[1], [0]]) :    #  |0⟩
  q3 := Matrix([[1], [0]]) :    #  |0⟩
  Co0 := K(K(q1, q2), q3);
  St := Transpose(VSte(n)) :
  |V0⟩ := Multiply(T(Co0), St)[1, 1];
```

$$Co0 := \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ \beta \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|V0\rangle := \alpha\,|000\rangle + \beta\,|100\rangle \tag{14}$$

$$\begin{aligned} \mathbf{V1} &= \mathbf{M1} \cdot \mathbf{V0} \\ &= \mathbf{M1}(\alpha|\mathbf{000}\rangle + \beta|\mathbf{100}\rangle) \\ &= \alpha|\mathbf{000}\rangle + \beta|\mathbf{110}\rangle \\ &= \alpha|\mathbf{0}\rangle + \beta|\mathbf{6}\rangle \end{aligned}$$

```
> Co1 := Multiply(M1, Co0);
  `|V1⟩` := Multiply(T(Co1), St)[1, 1];
```

$$Co1 := \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \beta \\ 0 \end{bmatrix}$$

$$|V1\rangle := \alpha\,|000\rangle + \beta\,|110\rangle \tag{15}$$

$$
\begin{aligned}
\mathbf{V2} &= \mathbf{M2}\cdot\mathbf{V1}\\
&= \mathbf{M2}(\alpha|\mathbf{000}\rangle + \beta|\mathbf{110}\rangle)\\
&= \alpha|\mathbf{000}\rangle + \beta|\mathbf{111}\rangle\\
&= \alpha|\mathbf{0}\rangle + \beta|\mathbf{7}\rangle\\
&= \alpha|\mathbf{0}\rangle|\mathbf{0}\rangle|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle|\mathbf{1}\rangle|\mathbf{1}\rangle
\end{aligned}
$$

```
> Co2 := Multiply(M2, Co1);
  `|V2⟩` := Multiply(T(Co2), St)[1, 1];
```

$$Co2 := \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \beta \end{bmatrix}$$

$$|V2\rangle := \alpha\,|000\rangle + \beta\,|111\rangle \tag{16}$$

The Hadamard H3 matrix/operator

$$\mathrm{H}\,|\mathbf{0}\rangle = \frac{\alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle}{\sqrt{2}};\ \mathrm{H}\,|\mathbf{1}\rangle = \frac{\alpha|\mathbf{0}\rangle - \beta|\mathbf{1}\rangle}{\sqrt{2}},$$

$$\mathbf{V3} = \mathbf{H3}\cdot\mathbf{V2}$$

$$= \alpha\left(\mathbf{H}|\mathbf{0}\rangle\otimes\mathbf{H}|\mathbf{0}\rangle\otimes\mathbf{H}|\mathbf{0}\rangle\right) + \beta\left(\mathbf{H}|\mathbf{1}\rangle\otimes\mathbf{H}|\mathbf{1}\rangle\otimes\mathbf{H}|\mathbf{1}\rangle\right)$$

$$\mathbf{V3} = \alpha\left(\frac{|\mathbf{0}\rangle+|\mathbf{1}\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|\mathbf{0}\rangle+|\mathbf{1}\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|\mathbf{0}\rangle+|\mathbf{1}\rangle}{\sqrt{2}}\right)$$

$$+\,\beta\otimes\left(\frac{|\mathbf{0}\rangle-|\mathbf{1}\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|\mathbf{0}\rangle-|\mathbf{1}\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|\mathbf{0}\rangle-|\mathbf{1}\rangle}{\sqrt{2}}\right)$$

$$\mathbf{V3} = \alpha|\mathbf{000}\rangle + \alpha|\mathbf{001}\rangle + \alpha|\mathbf{010}\rangle + \alpha|\mathbf{011}\rangle + \alpha|\mathbf{100}\rangle + \alpha|\mathbf{101}\rangle + \alpha|\mathbf{110}\rangle + \alpha|\mathbf{111}\rangle$$
$$+\ \beta|\mathbf{000}\rangle - \beta|\mathbf{001}\rangle - \beta|\mathbf{010}\rangle + \beta|\mathbf{011}\rangle - \beta|\mathbf{100}\rangle + \beta|\mathbf{101}\rangle + \beta|\mathbf{110}\rangle - \beta|\mathbf{111}\rangle$$

$$\mathbf{V3} = (\alpha+\beta)|0\rangle + (\alpha-\beta)|1\rangle + (\alpha-\beta)|2\rangle + (\alpha+\beta)|3\rangle + (\alpha-\beta)|4\rangle + (\alpha+\beta)|5\rangle + (\alpha+\beta)6\rangle + (\alpha-\beta)|7\rangle$$

$> \; Co3 := simplify\left(\sqrt{8}\; Multiply(H3, Co2)\right);$
$`|V3\rangle` := Multiply(T(Co3), St)[1, 1];$

$$Co3 := \begin{bmatrix} \alpha+\beta \\ \alpha-\beta \\ \alpha-\beta \\ \alpha+\beta \\ \alpha-\beta \\ \alpha+\beta \\ \alpha+\beta \\ \alpha-\beta \end{bmatrix}$$

$$|V3\rangle := (\alpha+\beta)\;|000\rangle + (\alpha-\beta)\;|001\rangle + (\alpha-\beta)\;|010\rangle + (\alpha+\beta)\;|011\rangle + (\alpha-\beta)\;|100\rangle + (\alpha+\beta)\;|101\rangle + (\alpha+\beta)\;|110\rangle + (\alpha-\beta)\;|111\rangle \qquad \textbf{(17)}$$

9-qubit code $\Rightarrow 2^9$ states. Large one column matrix: 512 by 1

$$|000000000\rangle \rightarrow |0\rangle$$
$$|000000001\rangle \rightarrow |1\rangle$$
$$\downarrow$$
$$|111111111\rangle \rightarrow |511\rangle \text{ or octal } |777\rangle_o$$

$$\mathbf{V4} = \alpha\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\Big|00\Big\rangle\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\Big|00\Big\rangle\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\Big|00\Big\rangle$$
$$+\;\beta\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\Big|00\Big\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\Big|00\Big\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\Big|00\Big\rangle$$

$$\mathbf{V4} = (\alpha+\beta)|000000000\rangle + (\alpha-\beta)|000000100\rangle + (\alpha-\beta)|000100000\rangle + (\alpha+\beta)|000100100\rangle$$
$$+ (\alpha-\beta)|100000000\rangle + (\alpha+\beta)|100000100\rangle + (\alpha+\beta)|100100000\rangle + (\alpha-\beta)|100100100\rangle$$

$$\mathbf{V4} = (\alpha+\beta)|000\rangle_o + (\alpha-\beta)|004\rangle_o + (\alpha-\beta)|040\rangle_o + (\alpha+\beta)|044\rangle_o$$
$$+ (\alpha-\beta)|400\rangle_o + (\alpha+\beta)|404\rangle_o + (\alpha+\beta)|440\rangle_o + (\alpha-\beta)|444\rangle_o$$

$$\mathbf{V4} = (\alpha+\beta)|0\rangle + (\alpha-\beta)|4\rangle + (\alpha-\beta)|32\rangle + (\alpha+\beta)|36\rangle$$
$$+ (\alpha-\beta)|256\rangle + (\alpha+\beta)|260\rangle + (\alpha+\beta)|288\rangle + (\alpha-\beta)|292\rangle$$

```
> n = 9 :
qa := 1/sqrt(2) Matrix([[1], [1]]) :      #  |0⟩ + |1⟩
qb := 1/sqrt(2) Matrix([[1], [-1]]) :     #  |0⟩ - |1⟩
q2 := Matrix([[1], [0]]) :                #  |0⟩
q3 := Matrix([[1], [0]]) :                #  |0⟩

A := α·K(K(qa, K(q2, q3)), K(K(qa, K(q2, q3)), K(qa, K(q2, q3)))) :
B := β·K(K(qb, K(q2, q3)), K(K(qb, K(q2, q3)), K(qb, K(q2, q3)))) :
Co4 := simplify(sqrt(8) (A + B));
Vr(Co4, 3);
St := Transpose(VSte(9)) :
`|V4⟩` := Multiply(T(Co4), St)[1, 1];
```

$$Co4 := \begin{bmatrix} \text{512 x 1 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix}$$

$$(\alpha+\beta)_{|000\rangle, |000\rangle, |000\rangle}$$

$$(\alpha-\beta)_{|004\rangle, |004\rangle, |100\rangle}$$

$$(\alpha-\beta)_{|032\rangle, |040\rangle, |000\rangle}$$

$$(\alpha+\beta)_{|036\rangle, |044\rangle, |100\rangle}$$

$$(\alpha-\beta)_{|256\rangle, |400\rangle, |000\rangle}$$

$$(\alpha+\beta)_{|260\rangle, |404\rangle, |100\rangle}$$

$$(\alpha+\beta)_{|288\rangle, |440\rangle, |000\rangle}$$

$$(\alpha-\beta)_{|292\rangle, |444\rangle, |100\rangle}$$

$$|V4\rangle := (\alpha+\beta)\ |000000000\rangle + (\alpha-\beta)\ |000000100\rangle + (\alpha-\beta)\ |000100000\rangle + (\alpha+\beta)\ |000100100\rangle + (\alpha-\beta)\ |100000000\rangle + (\alpha+\beta)\ |100000100\rangle + (\alpha+\beta)\ |100100000\rangle + (\alpha-\beta)\ |100100100\rangle \quad \textbf{(18)}$$

$$\mathbf{V5 = M3 \cdot V4}$$
$$\mathbf{= (\alpha+\beta)|000\rangle_o + (\alpha-\beta)|006\rangle_o + (\alpha-\beta)|060\rangle_o + (\alpha+\beta)|066\rangle_o}$$
$$\mathbf{+ (\alpha-\beta)|600\rangle_o + (\alpha+\beta)|606\rangle_o + (\alpha+\beta)|660\rangle_o + (\alpha-\beta)|666\rangle_o}$$

> $Co5 := Multiply(M3, Co4);\ Vr(Co5, 3);\ |V5\rangle := Multiply(T(Co5), St)[1, 1];$

$$Co5 := \begin{bmatrix} \text{512 x 1 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix}$$

$$(\alpha+\beta)_{|000\rangle,\ |000\rangle,\ |000000000\rangle}$$
$$(\alpha-\beta)_{|006\rangle,\ |006\rangle,\ |000000110\rangle}$$
$$(\alpha-\beta)_{|048\rangle,\ |060\rangle,\ |000110000\rangle}$$
$$(\alpha+\beta)_{|054\rangle,\ |066\rangle,\ |000110110\rangle}$$
$$(\alpha-\beta)_{|384\rangle,\ |600\rangle,\ |110000000\rangle}$$
$$(\alpha+\beta)_{|390\rangle,\ |606\rangle,\ |110000110\rangle}$$
$$(\alpha+\beta)_{|432\rangle,\ |660\rangle,\ |110110000\rangle}$$
$$(\alpha-\beta)_{|438\rangle,\ |666\rangle,\ |110110110\rangle}$$

$$|V5\rangle := (\alpha+\beta)\ |110000110\rangle + (\alpha+\beta)\ |110110000\rangle + (\alpha-\beta)\ |110110110\rangle + (\alpha-\beta)\ |000000110\rangle + (\alpha-\beta)\ |000110000\rangle + (\alpha+\beta)\ |000110110\rangle + (\alpha-\beta)\ |110000000\rangle + (\alpha+\beta)\ |000000000\rangle \quad \textbf{(19)}$$

$$\mathbf{V6 = M4 \cdot V5}$$
$$\mathbf{= (\alpha+\beta)|000\rangle_o + (\alpha-\beta)|007\rangle_o + (\alpha-\beta)|070\rangle_o + (\alpha+\beta)|077\rangle_o}$$
$$\mathbf{+ (\alpha-\beta)|700\rangle_o + (\alpha+\beta)|707\rangle_o + (\alpha+\beta)|770\rangle_o + (\alpha-\beta)|777\rangle_o}$$

> $Co6 := Multiply(M4, Co5); Vr(Co6, 3);\ `|V6\rangle` := Multiply(T(Co6), St)[1, 1];$

$$Co6 := \begin{bmatrix} \text{512 x 1 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix}$$

$$(\alpha+\beta)_{|000\rangle,\ |000\rangle,\ |000000000\rangle}$$
$$(\alpha-\beta)_{|007\rangle,\ |007\rangle,\ |000000111\rangle}$$
$$(\alpha-\beta)_{|056\rangle,\ |070\rangle,\ |000111000\rangle}$$
$$(\alpha+\beta)_{|063\rangle,\ |077\rangle,\ |000111111\rangle}$$
$$(\alpha-\beta)_{|448\rangle,\ |700\rangle,\ |111000000\rangle}$$
$$(\alpha+\beta)_{|455\rangle,\ |707\rangle,\ |111000111\rangle}$$
$$(\alpha+\beta)_{|504\rangle,\ |770\rangle,\ |111111000\rangle}$$
$$(\alpha-\beta)_{|511\rangle,\ |777\rangle,\ |111111111\rangle}$$

$$|V6\rangle := (\alpha+\beta)\ |000000000\rangle + (\alpha-\beta)\ |000000111\rangle + (\alpha-\beta)\ |000111000\rangle + (\alpha+\beta)\ |000111111\rangle + (\alpha-\beta)\ |111000000\rangle + (\alpha+\beta)\ |111000111\rangle + (\alpha+\beta)\ |111111000\rangle + (\alpha-\beta)\ |111111111\rangle \quad \textbf{(20)}$$

Qubit Error: QPF

A phase-flip error on the first qubit

$$\alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle \rightarrow \alpha|\mathbf{0}\rangle - \beta|\mathbf{1}\rangle \qquad \textbf{or}$$

$$\alpha|+\rangle + \beta|-\rangle \rightarrow \alpha|-\rangle + \beta|+\rangle$$

A qubit-flip error on the first qubit

$$\alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle \rightarrow \alpha|\mathbf{1}\rangle + \beta|\mathbf{0}\rangle$$

Both types of errors

$$\alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle \rightarrow \alpha|\mathbf{1}\rangle - \beta|\mathbf{0}\rangle$$

$$\boldsymbol{\alpha}\left(\frac{|\mathbf{000}\rangle+|\mathbf{111}\rangle}{\sqrt{\mathbf{2}}}\right) \rightarrow \boldsymbol{\alpha}\left(\frac{|\mathbf{100}\rangle-|\mathbf{011}\rangle}{\sqrt{\mathbf{2}}}\right)$$

$$\boldsymbol{\beta}\left(\frac{|\mathbf{000}\rangle-|\mathbf{111}\rangle}{\sqrt{\mathbf{2}}}\right) \rightarrow \boldsymbol{\beta}\left(\frac{|\mathbf{100}\rangle+|\mathbf{011}\rangle}{\sqrt{\mathbf{2}}}\right)$$

if qp = 1

$$V7 = QPF \cdot V6$$

$$= (\alpha + \beta)|400\rangle_o + (\alpha - \beta)|407\rangle_o + (\alpha - \beta)|470\rangle_o + (\alpha + \beta)|477\rangle_o$$
$$+ (-\alpha + \beta)|300\rangle_o + (-\alpha - \beta)|307\rangle_o - (\alpha - \beta)|370\rangle_o + (-\alpha + \beta)|377\rangle_o$$

```
> Co7 := Multiply(QPF, Co6);
  Vr(Co7, 3);
  `|V7〉` := Multiply(T(Co7), St)[1, 1];
```

$$Co7 := \begin{bmatrix} \textit{512 x 1 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix}$$

$$(-\alpha + \beta)_{|192\rangle, |300\rangle, |011000000\rangle}$$
$$(-\alpha - \beta)_{|199\rangle, |307\rangle, |011000111\rangle}$$
$$(-\alpha - \beta)_{|248\rangle, |370\rangle, |011111000\rangle}$$
$$(-\alpha + \beta)_{|255\rangle, |377\rangle, |011111111\rangle}$$
$$(\alpha + \beta)_{|256\rangle, |400\rangle, |100000000\rangle}$$
$$(\alpha - \beta)_{|263\rangle, |407\rangle, |100000111\rangle}$$
$$(\alpha - \beta)_{|312\rangle, |470\rangle, |100111000\rangle}$$
$$(\alpha + \beta)_{|319\rangle, |477\rangle, |100111111\rangle}$$

$$|V7\rangle := (-\alpha + \beta)\ |011000000\rangle + (-\alpha - \beta)\ |011000111\rangle + (-\alpha - \beta)\ |011111000\rangle + (-\alpha + \beta)\ |011111111\rangle + (\alpha + \beta)\ |100000000\rangle + (\alpha - \beta)\ |100000111\rangle + (\alpha - \beta)\ |100111000\rangle + (\alpha + \beta)\ |100111111\rangle \qquad \textbf{(21)}$$

if qp = 1

$V8 = M4 \cdot V7$

$= (\alpha + \beta)|500\rangle_o + (\alpha - \beta)|506\rangle_o + (\alpha - \beta)|560\rangle_o + (\alpha + \beta)|566\rangle_o$
$+ (-\alpha + \beta)|300\rangle_o + (-\alpha - \beta)|306\rangle_o + (-\alpha - \beta)|360\rangle_o + (-\alpha + \beta)|366\rangle_o$

```
> Co8 := Multiply(M4, Co7);
  Vr(Co8, 3);
  `|V8⟩` := Multiply(T(Co8), St)[1, 1];
```

$$Co8 := \begin{bmatrix} \text{512 x 1 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix}$$

$$(-\alpha + \beta)_{|192\rangle, |300\rangle, |011000000\rangle}$$
$$(-\alpha - \beta)_{|198\rangle, |306\rangle, |011000110\rangle}$$
$$(-\alpha - \beta)_{|240\rangle, |360\rangle, |011110000\rangle}$$
$$(-\alpha + \beta)_{|246\rangle, |366\rangle, |011110110\rangle}$$
$$(\alpha + \beta)_{|320\rangle, |500\rangle, |101000000\rangle}$$
$$(\alpha - \beta)_{|326\rangle, |506\rangle, |101000110\rangle}$$
$$(\alpha - \beta)_{|368\rangle, |560\rangle, |101110000\rangle}$$
$$(\alpha + \beta)_{|374\rangle, |566\rangle, |101110110\rangle}$$

$$|V8\rangle := (-\alpha - \beta)\ |011000110\rangle + (-\alpha - \beta)\ |011110000\rangle + (-\alpha + \beta)\ |011110110\rangle + (\alpha + \beta)\ |101000000\rangle + (\alpha - \beta)\ |101000110\rangle + (\alpha - \beta)\ |101110000\rangle + (\alpha + \beta)\ |101110110\rangle + (-\alpha + \beta)\ |011000000\rangle \quad \textbf{(22)}$$

if qp = 1

$V9 = M3 \cdot V8$

$= (\alpha + \beta)|700\rangle_o + (\alpha - \beta)|704\rangle_o + (\alpha - \beta)|740\rangle_o + (\alpha + \beta)|744\rangle_o$
$+ (-\alpha + \beta)|300\rangle_o + (-\alpha - \beta)|304\rangle_o + (-\alpha - \beta)|340\rangle_o + (\alpha - \beta)|344\rangle_o$

> Co9 := Multiply(M3, Co8) : Vr(Co9, 3); `|V9⟩` := Multiply(T(Co9), St)[1, 1];

$$Co9 := \left[\begin{array}{c} \text{512 x 1 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{array}\right]$$

$$(-\alpha+\beta)_{|192\rangle,\ |300\rangle,\ |011000000\rangle}$$
$$(-\alpha-\beta)_{|196\rangle,\ |304\rangle,\ |011000100\rangle}$$
$$(-\alpha-\beta)_{|224\rangle,\ |340\rangle,\ |011100000\rangle}$$
$$(-\alpha+\beta)_{|228\rangle,\ |344\rangle,\ |011100100\rangle}$$
$$(\alpha+\beta)_{|448\rangle,\ |700\rangle,\ |111000000\rangle}$$
$$(\alpha-\beta)_{|452\rangle,\ |704\rangle,\ |111000100\rangle}$$
$$(\alpha-\beta)_{|480\rangle,\ |740\rangle,\ |111100000\rangle}$$
$$(\alpha+\beta)_{|484\rangle,\ |744\rangle,\ |111100100\rangle}$$

$$|V9\rangle := (\alpha+\beta)\ |111100100\rangle + (-\alpha-\beta)\ |011100000\rangle + (-\alpha+\beta)\ |011100100\rangle + (\alpha+\beta)\ |111000000\rangle + (\alpha-\beta)\ |111000100\rangle + (\alpha-\beta)\ |111100000\rangle + (-\alpha-\beta)\ |011000100\rangle + (-\alpha+\beta)\ |011000000\rangle \quad \textbf{(23)}$$

if qp = 1

$$V10 = M5 \cdot V9 = (\alpha+\beta)|300\rangle_o + (\alpha-\beta)|304\rangle_o + (\alpha-\beta)|340\rangle_o + (\alpha+\beta)|344\rangle_o + (-\alpha+\beta)|700\rangle_o + (-\alpha-\beta)|704\rangle_o + (-\alpha-\beta)|740\rangle_o + (-\alpha+\beta)|744\rangle_o$$

> Co10 := Multiply(M5, Co9); Vr(Co10, 3); `|V10⟩` := Multiply(T(Co10), St)[1, 1];

$$(\alpha+\beta)_{|192\rangle,\ |300\rangle,\ |011000000\rangle}$$
$$(\alpha-\beta)_{|196\rangle,\ |304\rangle,\ |011000100\rangle}$$
$$(\alpha-\beta)_{|224\rangle,\ |340\rangle,\ |011100000\rangle}$$
$$(\alpha+\beta)_{|228\rangle,\ |344\rangle,\ |011100100\rangle}$$
$$(-\alpha+\beta)_{|448\rangle,\ |700\rangle,\ |111000000\rangle}$$
$$(-\alpha-\beta)_{|452\rangle,\ |704\rangle,\ |111000100\rangle}$$
$$(-\alpha-\beta)_{|480\rangle,\ |740\rangle,\ |111100000\rangle}$$
$$(-\alpha+\beta)_{|484\rangle,\ |744\rangle,\ |111100100\rangle}$$

$$|V10\rangle := (\alpha+\beta)\ |011000000\rangle + (\alpha-\beta)\ |011000100\rangle + (\alpha-\beta)\ |011100000\rangle + (\alpha+\beta)\ |011100100\rangle + (-\alpha+\beta)\ |111000000\rangle + (-\alpha-\beta)\ |111000100\rangle + (-\alpha-\beta)\ |111100000\rangle + (-\alpha+\beta)\ |111100100\rangle \quad \textbf{(24)}$$

if qp = 1

$$V10 = \alpha\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\left|11\right\rangle\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle$$
$$+\ \beta\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left|11\right\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\left|00\right\rangle$$

$$= \alpha(q1 \otimes|11\rangle\otimes q4 \otimes|00\rangle\otimes q7 \otimes|00\rangle) + \beta(q1 \otimes|11\rangle\otimes q4 \otimes|00\rangle\otimes q7 \otimes|00\rangle)$$

Now working with the 1^{st}, 4^{th} & 7^{th} qubits

$$V10 = \alpha\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)+\beta\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\Big\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

Apply the H3 operator

$$V11 = H3 \cdot V10$$
$$= \alpha|100\rangle+\beta|011\rangle$$
$$= \alpha|4\rangle+\beta|3\rangle$$

```
> if qp = 1 then
    simplify( (α·K(qb, K(qa, qa)) + β·K(qa, K(qb, qb))) ) :
  elif qp = 2 then
    simplify( (α·K(qa, K(qb, qa)) + β·K(qb, K(qa, qb))) ) :
  elif qp = 3 then
    simplify( (α·K(qa, K(qa, qb)) + β·K(qb, K(qb, qa))) ) :
 end if :
 Co11 := simplify(Multiply(H3, %));
 St := Transpose(VSte(3)) :
 `|V11⟩` := Multiply(T(Co11), St)[1, 1];
```

$$Co11 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \beta \\ \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|V11\rangle := \beta\,|011\rangle + \alpha\,|100\rangle \qquad (25)$$

if qp = 1

$$
\begin{aligned}
V12 &= M2 \cdot V11\\
&= M2(\alpha|100\rangle+\beta|011\rangle\\
&= \alpha|101\rangle+\beta|011\rangle\\
&= \alpha|5\rangle+\beta|3\rangle
\end{aligned}
$$

```
> Co12 := Multiply(M2, Co11);
  `|V12⟩` := Multiply(T(Co12), St)[1, 1];
```

$$
Co12 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \beta \\ 0 \\ \alpha \\ 0 \\ 0 \end{bmatrix}
$$

$$
|V12\rangle := \beta\, |011\rangle + \alpha\, |101\rangle \tag{26}
$$

if qp = 1

$$
\begin{aligned}
V13 &= M1 \cdot V12\\
&= M1(\alpha|101\rangle+\beta|011\rangle\\
&= \alpha|111\rangle+\beta|011\rangle\\
&= \alpha|7\rangle+\beta|3\rangle
\end{aligned}
$$

```
> Co13 := Multiply(M1, Co12);
  `|V13⟩` := Multiply(T(Co13), St)[1, 1];
```

$$
Co13 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \beta \\ 0 \\ 0 \\ 0 \\ \alpha \end{bmatrix}
$$

$$
|V13\rangle := \beta\, |011\rangle + \alpha\, |111\rangle \tag{27}
$$

if qp = 1

$V14 = M6 \cdot V13$

$= M6(\alpha|111\rangle + \beta|011\rangle)$

$= \alpha|011\rangle + \beta|111\rangle$

$= \alpha|3\rangle + \beta|7\rangle$

which can be re-written as

$V14 = (\alpha|0\rangle + \beta|1\rangle) \otimes |1\rangle \otimes |1\rangle$

if qp = 2 then $V14 = \alpha|010\rangle + \beta|110\rangle = \alpha|2\rangle + \beta|6\rangle$

if qp = 3 then $V14 = \alpha|001\rangle + \beta|101\rangle = \alpha|1\rangle + \beta|5\rangle$

```
> Co14 := Multiply(M6, Co13);
  `|V14〉` := Multiply(T(Co14), St)[1, 1];
```

$$Co14 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha \\ 0 \\ 0 \\ 0 \\ \beta \end{bmatrix}$$

$$|V14\rangle := \alpha\,|011\rangle + \beta\,|111\rangle \tag{28}$$

Compare to V0

$\mathbf{V0} = (\alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle) \otimes |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle$

$= \alpha|\mathbf{000}\rangle + \beta|\mathbf{100}\rangle$

$= \alpha|\mathbf{0}\rangle + \beta|\mathbf{4}\rangle$

```
> |V0〉 := Multiply(T(Co0), St)[1, 1];
```

$$|V0\rangle := \alpha\,|000\rangle + \beta\,|100\rangle \tag{29}$$