```
> restart;
> interface(warnlevel = 0) :   #   Maple 12
> with(LinearAlgebra) :
```

Two-Level Composition of a 3 by 3 Matrix

Two-Level Matrix Composition of a 3 by 3 Matrix

The goal is to generate a set of matrices such that V_n--$V_2V_1U = I$. The V_i matrices are known as two-level matrices.We will start with a 3 by 3 unitary matrix U to illustrate the procedure and to show that these two-level matrices are simple rotation operators/matrices.

```
> U := RowOperation(IdentityMatrix(3), [1, 2]);
  I3 := Multiply(U, HermitianTranspose(U));
```

$$U := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.1)$$

```
> a := U[1, 1];  # the starting point
  b := U[2, 1];
  α := conjugate(a);
  β := conjugate(b);
  r := sqrt(|a|^2 + |b|^2);
  V1 := Matrix([[α/r, β/r, 0], [b/r, -a/r, 0], [0, 0, 1]]); # V1 is the first two-level matrix
  v1 := HermitianTranspose(V1);
```

$$a := 0$$
$$b := 1$$
$$\alpha := 0$$
$$\beta := 1$$
$$r := 1$$

$$V1 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v1 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.2)$$

Notice the form of V1

> $z := \sqrt{|x|^2 + |y|^2}$:
> $Matrix\left(\left[\left[\frac{x}{z}, \frac{y}{z}, 0\right], \left[\frac{y}{z}, -\frac{x}{z}, 0\right], [0, 0, 1]\right]\right)$;

$$\begin{bmatrix} \frac{x}{\sqrt{|x|^2+|y|^2}} & \frac{y}{\sqrt{|x|^2+|y|^2}} & 0 \\ \frac{y}{\sqrt{|x|^2+|y|^2}} & -\frac{x}{\sqrt{|x|^2+|y|^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.1.1}$$

> p1 := *Multiply*(*V1*, *U*);

$$p1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.3}$$

Notice that p1 equals the Identity Matrix, so we could stop at this point

> *Multiply*(p1, *Multiply*(*V1*, *U*));

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.2.1}$$

> *Multiply*(*HermitianTranspose*(*V1*), *HermitianTranspose*(*p1*));

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.2.2}$$

Our two-level matrix set consists of V1 and p1. Let's define V2 = p1, then our composition

$$V^{\dagger}_1 V^{\dagger}_2 = U$$

> a := *p1*[1, 1];
c := *p1*[3, 1];
$\mathscr{a}$:= *conjugate*(a);
$\mathscr{c}$:= *conjugate*(c);
$r := sqrt\left(|a|^2 + |c|^2\right);$

$$V2 := Matrix\left(\left[\left[\frac{\mathscr{a}}{r}, 0, \frac{\mathscr{c}}{r}\right], [0, 1, 0], \left[\frac{c}{r}, 0, -\frac{a}{r}\right]\right]\right);$$

v2 := *HermitianTranspose*(*V2*);

$$a := 1$$
$$c := 0$$
$$\mathscr{a} := 1$$
$$\mathscr{c} := 0$$
$$r := 1$$
$$V2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$v2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{1.4}$$

Notice the form of V2

> $\mathscr{z} := \sqrt{|x|^2 + |y|^2}$:

$$Matrix\left(\left[\left[\frac{x}{\mathscr{z}}, 0, \frac{y}{\mathscr{z}}\right], [0, 1, 0], \left[\frac{y}{\mathscr{z}}, 0, -\frac{x}{\mathscr{z}}\right]\right]\right);$$

$$\begin{bmatrix} \frac{x}{\sqrt{|x|^2 + |y|^2}} & 0 & \frac{y}{\sqrt{|x|^2 + |y|^2}} \\ 0 & 1 & 0 \\ \frac{y}{\sqrt{|x|^2 + |y|^2}} & 0 & -\frac{x}{\sqrt{|x|^2 + |y|^2}} \end{bmatrix} \tag{1.3.1}$$

> $p2 := Multiply(V2, p1);$

$$p2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (1.5)$$

> $V3 := HermitianTranspose(p2);$
> $v3 := HermitianTranspose(V3);$

$$V3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$v3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (1.6)$$

> $Multiply(V3, Multiply(V2, Multiply(V1, U)));$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.7)$$

> $\mathcal{U} := Multiply(v1, Multiply(v2, v3));$ # $the\ \ two-level\ \ matrix\ \ set$

$$\mathcal{U} := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.8)$$

> $'\mathcal{U} = U', Equal(U, \mathcal{U});$

$$\mathcal{U} = U, true \quad (1.9)$$

Two-Level Composition of a 4 by 4 Matrix

Two-Level Composition of a 4 by 4 Matrix

> $U := RowOperation(IdentityMatrix(4), [1, 2]) : U := RowOperation(U, [3, 4]);$
> $I4 := Multiply(U, HermitianTranspose(U));$

$$U := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$I4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.1)$$

> a := $U[1, 1]$; b := $U[2, 1]$; $\alpha := conjugate(a)$; $\beta := conjugate(b)$; $r := sqrt(|a|^2 + |b|^2)$;
> $V1 := Matrix\left(\left[\left[\frac{\alpha}{r}, \frac{\beta}{r}, 0, 0\right], \left[\frac{b}{r}, -\frac{a}{r}, 0, 0\right], [0, 0, 1, 0], [0, 0, 0, 1]\right]\right)$;
> $v1 := HermitianTranspose(V1)$;

$$a := 0$$
$$b := 1$$
$$\alpha := 0$$
$$\beta := 1$$
$$r := 1$$
$$V1 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$v1 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.2)$$

> $p1 := Multiply(V1, U)$;

$$p1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (2.3)$$

Notice that p1 has the proper form for a 4 by 4 matrix, so we could stop at this point

> $Matrix([[I2, 0], [0, M]])$; # "proper form"

$$\begin{bmatrix} I\,2 & 0 \\ 0 & M \end{bmatrix} \quad (2.1.1)$$

> $Multiply(p1, Multiply(V1, U))$;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.1.2)$$

> $Multiply(v1, HermitianTranspose(p1))$; # two− level set

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (2.1.3)$$

> $a := p1[1,1];\ c := p1[3,1];\ \mathscr{a} := conjugate(a);\ \mathscr{c} := conjugate(c);\ r := sqrt(|a|^2 + |c|^2);$
> $V2 := Matrix\left(\left[\left[\frac{\mathscr{a}}{r}, 0, \frac{\mathscr{c}}{r}, 0\right], [0, 1, 0, 0], \left[\frac{c}{r}, 0, -\frac{a}{r}, 0\right], [0, 0, 0, 1]\right]\right);$
> $v2 := HermitianTranspose(V2);\ p2 := Multiply(V2, p1);$

$$a := 1$$
$$c := 0$$
$$\mathscr{a} := 1$$
$$\mathscr{c} := 0$$
$$r := 1$$
$$V2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$v2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$p2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \textbf{(2.4)}$$

> $a := p2[1,1];\ d := p2[4,1];\ \mathscr{a} := conjugate(a);\ \mathscr{d} := conjugate(d);\ r := sqrt(|a|^2 + |d|^2);$
> $V3 := Matrix\left(\left[\left[\frac{\mathscr{a}}{r}, 0, 0, \frac{\mathscr{d}}{r}, \right], [0, 1, 0, 0], [0, 0, 1, 0], \left[\frac{d}{r}, 0, 0, -\frac{a}{r}\right]\right]\right);$
> $v3 := HermitianTranspose(V3);$

$$a := 1$$
$$d := 0$$
$$\mathscr{a} := 1$$
$$\mathscr{d} := 0$$
$$r := 1$$
$$V3 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(2.5)

$$v3 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.5}$$

> $p3 := Multiply(V3, p2);$

$$p3 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (2.6)$$

Second set of operations on the 3 by 3 submatrix

> $U2 := SubMatrix(p3, [2..4], [2..4]);$
> $Multiply(HermitianTranspose(U2), U2);$

$$U2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.7)$$

> $\mathrm{a} := p3[2, 2]; \mathrm{b} := p3[3, 2]; \mathscr{a} := conjugate(\mathrm{a}); \mathscr{b} := conjugate(\mathrm{b}); r := sqrt(|\mathrm{a}|^2 + |\mathrm{b}|^2);$
> $\mathrm{V4} := Matrix\left(\left[[1, 0, 0, 0], \left[0, \frac{\mathscr{a}}{r}, \frac{\mathscr{b}}{r}, 0\right], \left[0, \frac{\mathrm{b}}{r}, -\frac{\mathrm{a}}{r}, 0\right], [0, 0, 0, 1]\right]\right);$
> $v4 := HermitianTranspose(V4); p4 := Multiply(V4, p3);$

$$a := 1$$
$$b := 0$$
$$\mathscr{a} := 1$$
$$\mathscr{b} := 0$$
$$r := 1$$

$$V4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$v4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (2.8)$$

> a := $p4[2, 2]$; c := $p4[4, 2]$; $\mathcal{a}$:= $conjugate(\mathrm{a})$; $\mathcal{c}$:= $conjugate(\mathrm{c})$; $r := sqrt\left(|\mathrm{a}|^2 + |\mathrm{c}|^2\right)$;
$V5 := Matrix\left(\left[[1, 0, 0, 0], \left[0, \frac{\mathcal{a}}{r}, 0, \frac{\mathcal{c}}{r}\right], [0, 0, 1, 0], \left[0, \frac{\mathrm{c}}{r}, 0, -\frac{\mathrm{a}}{r}\right]\right]\right)$;
$v5 := HermitianTranspose(V5)$; $p5 := Multiply(V5, p4)$;

$$a := 1$$

$$c := 0$$

$$\mathcal{a} := 1$$

$$\mathcal{c} := 0$$

$$r := 1$$

$$V5 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$v5 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$p5 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \textbf{(2.9)}$$

> $V6 := HermitianTranspose(p5)$;
$v6 := HermitianTranspose(V6)$;

$$V6 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$v6 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \textbf{(2.10)}$$

> $Multiply(V6, Multiply(V5, Multiply(V4, Multiply(V3, Multiply(V2, Multiply(V1, U))))))$;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(2.11)}$$

> $\mathcal{U} := Multiply(v1, Multiply(v2, Multiply(v3, Multiply(v4, Multiply(v5, v6)))));$

$$\mathcal{U} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.12}$$

> $'\mathcal{U} = U', Equal(U, \mathcal{U});$

$$\mathcal{U} = U, true \tag{2.13}$$

Another Two-Level Composition of a 4 by 4 Matrix

Two-Level Composition of a 4 by 4 Matrix

> $U := \frac{1}{2} \cdot Matrix([[1, 1, 1, 1], [1, I, -1, -I], [1, -1, 1, -1], [1, -I, -1, I]]);$
> $I4 := Multiply(U, HermitianTranspose(U));$

$$U := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\,I & -\frac{1}{2} & -\frac{1}{2}\,I \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\,I & -\frac{1}{2} & \frac{1}{2}\,I \end{bmatrix}$$

$$I4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.1}$$

> a := $U[1,1]$; b := $U[2,1]$; α := conjugate(a); β := conjugate(b); r := sqrt(|a|^2 + |b|^2);
V1 := simplify(Matrix([[$\frac{\alpha}{r}$, $\frac{\beta}{r}$, 0, 0], [$\frac{b}{r}$, $-\frac{a}{r}$, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]]));
v1 := simplify(HermitianTranspose(V1));

$$a := \frac{1}{2}$$

$$b := \frac{1}{2}$$

$$\alpha := \frac{1}{2}$$

$$\beta := \frac{1}{2}$$

$$r := \frac{1}{2}\sqrt{2}$$

$$V1 := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$v1 := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(3.2)}$$

> p1 := simplify(Multiply(V1, U));

$$p1 := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{4}\sqrt{2} + \frac{1}{4}\,\mathrm{I}\,\sqrt{2} & 0 & \frac{1}{4}\sqrt{2} - \frac{1}{4}\,\mathrm{I}\,\sqrt{2} \\ 0 & \frac{1}{4}\sqrt{2} - \frac{1}{4}\,\mathrm{I}\,\sqrt{2} & \frac{1}{2}\sqrt{2} & \frac{1}{4}\sqrt{2} + \frac{1}{4}\,\mathrm{I}\,\sqrt{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\,\mathrm{I} & -\frac{1}{2} & \frac{1}{2}\,\mathrm{I} \end{bmatrix} \qquad \textbf{(3.3)}$$

> a := *p1*[1, 1];
c := *p1*[3, 1];
$\bar{a}$:= *conjugate*(a);
$\bar{c}$:= *conjugate*(c);
r := *sqrt*(|a|^2 + |c|^2);
V2 := *simplify*(*Matrix*([[$\frac{\bar{a}}{r}$, 0, $\frac{\bar{c}}{r}$, 0], [0, 1, 0, 0], [$\frac{c}{r}$, 0, $-\frac{a}{r}$, 0], [0, 0, 0, 1]]));
v2 := *simplify*(*HermitianTranspose*(*V2*));

$$a := \frac{1}{2}\sqrt{2}$$

$$c := \frac{1}{2}$$

$$\bar{a} := \frac{1}{2}\sqrt{2}$$

$$\bar{c} := \frac{1}{2}$$

$$r := \frac{1}{2}\sqrt{3}$$

$$V2 := \begin{bmatrix} \frac{1}{3}\sqrt{2}\sqrt{3} & 0 & \frac{1}{3}\sqrt{3} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3}\sqrt{3} & 0 & -\frac{1}{3}\sqrt{2}\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$v2 := \begin{bmatrix} \frac{1}{3}\sqrt{2}\sqrt{3} & 0 & \frac{1}{3}\sqrt{3} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3}\sqrt{3} & 0 & -\frac{1}{3}\sqrt{2}\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.4}$$

> *p2* := *simplify*(*Multiply*(*V2*, *p1*));

$$p2 := \begin{bmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{6}\,\mathrm{I}\sqrt{3} & \frac{1}{6}\sqrt{3} & -\frac{1}{6}\,\mathrm{I}\sqrt{3} \\ 0 & \frac{1}{4}\sqrt{2} - \frac{1}{4}\,\mathrm{I}\sqrt{2} & \frac{1}{2}\sqrt{2} & \frac{1}{4}\sqrt{2} + \frac{1}{4}\,\mathrm{I}\sqrt{2} \\ 0 & \frac{1}{4}\sqrt{2}\sqrt{3} + \frac{1}{12}\,\mathrm{I}\sqrt{3}\sqrt{2} & -\frac{1}{6}\sqrt{2}\sqrt{3} & \frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{12}\,\mathrm{I}\sqrt{3}\sqrt{2} \\ \frac{1}{2} & -\frac{1}{2}\,\mathrm{I} & -\frac{1}{2} & \frac{1}{2}\,\mathrm{I} \end{bmatrix} \tag{3.5}$$

> a := p2[1, 1];
d := p2[4, 1];
a := conjugate(a);
d := conjugate(d);
$r := sqrt(|a|^2 + |d|^2)$;
$V3 := simplify\left(Matrix\left(\left[\left[\frac{a}{r}, 0, 0, \frac{d}{r}\right], [0, 1, 0, 0], [0, 0, 1, 0], \left[\frac{d}{r}, 0, 0, -\frac{a}{r}\right]\right]\right)\right);$
$v3 := simplify(HermitianTranspose(V3));$

$$a := \frac{1}{2}\sqrt{3}$$

$$d := \frac{1}{2}$$

$$a := \frac{1}{2}\sqrt{3}$$

$$d := \frac{1}{2}$$

$$r := 1$$

$$V3 := \begin{bmatrix} \frac{1}{2}\sqrt{3} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2}\sqrt{3} \end{bmatrix}$$

$$v3 := \begin{bmatrix} \frac{1}{2}\sqrt{3} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2}\sqrt{3} \end{bmatrix} \quad \textbf{(3.6)}$$

> p3 := simplify(Multiply(V3, p2));

$$p3 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4}\sqrt{2} - \frac{1}{4} I\sqrt{2} & \frac{1}{2}\sqrt{2} & \frac{1}{4}\sqrt{2} + \frac{1}{4} I\sqrt{2} \\ 0 & \frac{1}{4}\sqrt{2}\sqrt{3} + \frac{1}{12} I\sqrt{3}\sqrt{2} & -\frac{1}{6}\sqrt{2}\sqrt{3} & \frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{12} I\sqrt{3}\sqrt{2} \\ 0 & \frac{1}{3} I\sqrt{3} & \frac{1}{3}\sqrt{3} & -\frac{1}{3} I\sqrt{3} \end{bmatrix} \quad \textbf{(3.7)}$$

Second set of operations on the 3 by 3 submatrix

> $U2 := simplify(SubMatrix(p3, [2..4], [2..4]));$
> $simplify(Multiply(HermitianTranspose(U2), U2));$

$$U2 := \begin{bmatrix} \frac{1}{4}\sqrt{2} - \frac{1}{4}\,I\sqrt{2} & \frac{1}{2}\sqrt{2} & \frac{1}{4}\sqrt{2} + \frac{1}{4}\,I\sqrt{2} \\ \frac{1}{4}\sqrt{2}\sqrt{3} + \frac{1}{12}\,I\sqrt{3}\sqrt{2} & -\frac{1}{6}\sqrt{2}\sqrt{3} & \frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{12}\,I\sqrt{3}\sqrt{2} \\ \frac{1}{3}\,I\sqrt{3} & \frac{1}{3}\sqrt{3} & -\frac{1}{3}\,I\sqrt{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(3.8)}$$

> $a := p3[2,2];\ b := p3[3,2];\ \alpha := conjugate(a);\ \beta := conjugate(b);\ r := sqrt(|a|^2 + |b|^2);$
> $V4 := simplify\left(Matrix\left(\left[[1,0,0,0], \left[0, \frac{\alpha}{r}, \frac{\beta}{r}, 0\right], \left[0, \frac{b}{r}, -\frac{a}{r}, 0\right], [0,0,0,1]\right]\right)\right);$
> $v4 := simplify(HermitianTranspose(V4));$

$$a := \frac{1}{4}\sqrt{2} - \frac{1}{4}\,I\sqrt{2}$$

$$b := \frac{1}{4}\sqrt{2}\sqrt{3} + \frac{1}{12}\,I\sqrt{3}\sqrt{2}$$

$$\alpha := \frac{1}{4}\sqrt{2} + \frac{1}{4}\,I\sqrt{2}$$

$$\beta := \frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{12}\,I\sqrt{3}\sqrt{2}$$

$$r := \frac{1}{3}\sqrt{6}$$

$$V4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \left(\frac{1}{4} + \frac{1}{4}\,I\right)\sqrt{3} & \frac{3}{4} - \frac{1}{4}\,I & 0 \\ 0 & \frac{3}{4} + \frac{1}{4}\,I & \left(-\frac{1}{4} + \frac{1}{4}\,I\right)\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$v4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \left(\frac{1}{4} - \frac{1}{4}\,I\right)\sqrt{3} & \frac{3}{4} - \frac{1}{4}\,I & 0 \\ 0 & \frac{3}{4} + \frac{1}{4}\,I & \left(-\frac{1}{4} - \frac{1}{4}\,I\right)\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(3.9)}$$

> $p4 := simplify(Multiply(V4, p3));$

$$p4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{2}\sqrt{3} & \frac{1}{6}\,\mathrm{I}\sqrt{3}\sqrt{2} & \frac{1}{6}\sqrt{2}\sqrt{3} \\ 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\,\mathrm{I}\sqrt{2} \\ 0 & \frac{1}{3}\,\mathrm{I}\sqrt{3} & \frac{1}{3}\sqrt{3} & -\frac{1}{3}\,\mathrm{I}\sqrt{3} \end{bmatrix} \quad (3.10)$$

> $a := p4[2,2];\ c := p4[4,2];\ \mathcal{a} := conjugate(a);\mathcal{c} := conjugate(c);r := sqrt(|a|^2 + |c|^2);$
$V5 := simplify\left(Matrix\left(\left[[1,0,0,0], \left[0, \frac{\mathcal{a}}{r}, 0, \frac{\mathcal{c}}{r}\right], [0,0,1,0], \left[0, \frac{c}{r}, 0, -\frac{a}{r}\right]\right]\right)\right);$
$v5 := simplify(HermitianTranspose(V5));$

$$a := \frac{1}{3}\sqrt{2}\sqrt{3}$$

$$c := \frac{1}{3}\,\mathrm{I}\sqrt{3}$$

$$\mathcal{a} := \frac{1}{3}\sqrt{2}\sqrt{3}$$

$$\mathcal{c} := -\frac{1}{3}\,\mathrm{I}\sqrt{3}$$

$$r := 1$$

$$V5 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{2}\sqrt{3} & 0 & -\frac{1}{3}\,\mathrm{I}\sqrt{3} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3}\,\mathrm{I}\sqrt{3} & 0 & -\frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}$$

$$v5 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{2}\sqrt{3} & 0 & -\frac{1}{3}\,\mathrm{I}\sqrt{3} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3}\,\mathrm{I}\sqrt{3} & 0 & -\frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix} \quad (3.11)$$

> $p5 := simplify(Multiply(V5, p4));$ *# Notice we have the proper form for a 4 by 4 . Let's define V6 = p5*

$$p5 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\,\mathrm{I}\sqrt{2} \\ 0 & 0 & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\,\mathrm{I}\sqrt{2} \end{bmatrix} \quad (3.12)$$

> $V6 := simplify(HermitianTranspose(p5));$
> $v6 := simplify(HermitianTranspose(V6));$ # which is actually p5

$$V6 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ 0 & 0 & -\frac{1}{2}\,I\,\sqrt{2} & -\frac{1}{2}\,I\,\sqrt{2} \end{bmatrix}$$

$$v6 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\,I\,\sqrt{2} \\ 0 & 0 & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\,I\,\sqrt{2} \end{bmatrix} \tag{3.13}$$

> $simplify(Multiply(V6, Multiply(V5, Multiply(V4, Multiply(V3, Multiply(V2, Multiply(V1, U)))))));$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.14}$$

> $\mathcal{U} := simplify(Multiply(v1, Multiply(v2, Multiply(v3, Multiply(v4, Multiply(v5, v6))))));$

$$\mathcal{U} := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\,I & -\frac{1}{2} & -\frac{1}{2}\,I \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\,I & -\frac{1}{2} & \frac{1}{2}\,I \end{bmatrix} \tag{3.15}$$

> $'\mathcal{U} = U', Equal(U, \mathcal{U});$

$$\mathcal{U} = U, true \tag{3.16}$$