> *restart*;
> *interface*(*warnlevel* = 0) : # **Maple 12**
> *with*(*LinearAlgebra*) :

Chapter 7 Problem 5

> $TP := \mathbf{proc}(M1, M2)\ \mathbf{return}\ KroneckerProduct(M1, M2)\ \mathbf{end\ proc}$:

Defining matrices

> $\mathbf{I2} := IdentityMatrix(2);$
$\mathbf{X} := Matrix([[0, 1], [1, 0]]);$
$\mathbf{Y} := Matrix([[0, -i], [i, 0]]);$
$\mathbf{Z} := Matrix([[1, 0], [0, -1]]);$
$\mathbf{H} := \frac{1}{\sqrt{2}}(\mathbf{X} + \mathbf{Z}) :$
$'\sqrt{2} \cdot H' = \sqrt{2} \cdot \mathbf{H};$
$CNOT := RowOperation(IdentityMatrix(4), [3, 4]);$

$$I2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sqrt{2}\ H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$CNOT := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (1)$$

Bell's Matrices

> $Bell := Multiply(CNOT, TP(\mathbf{H}, \mathbf{I2}))$:
> $'\sqrt{2} \cdot Bell' = \sqrt{2} \cdot Bell$;
> $IBell := MatrixInverse(Bell)$:   # $(\mathbf{H} \otimes \mathbf{I2}) CNOT$
> $'\sqrt{2} \cdot IBell' = \sqrt{2} \cdot IBell$;

$$\sqrt{2}\ Bell = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\sqrt{2}\ IBell = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \quad \textbf{(2)}$$

Defining qubits in matrix form

$$|\mathbf{A}\rangle = \mathbf{a_0}|\mathbf{0}\rangle + \mathbf{a_1}|\mathbf{1}\rangle$$
$$|\mathbf{B}\rangle = \mathbf{b_0}|\mathbf{0}\rangle + \mathbf{b_1}|\mathbf{1}\rangle$$

$$|\mathbf{C1}\rangle = |\mathbf{A}\rangle \otimes |\mathbf{B}\rangle = (\mathbf{a_0}|\mathbf{0}\rangle + \mathbf{a_1}|\mathbf{1}\rangle) \otimes (\mathbf{a_0}|\mathbf{0}\rangle + \mathbf{b_1}|\mathbf{1}\rangle)$$

$$|\mathbf{C1}\rangle = \mathbf{a_0 b_0}|\mathbf{00}\rangle + \mathbf{a_0 b_1}|\mathbf{01}\rangle + \mathbf{a_1 b_0}|\mathbf{10}\rangle + \mathbf{a_1 b_1}|\mathbf{11}\rangle$$

> $A := Matrix([[a[0]], [a[1]]])$;
> $B := Matrix([[b[0]], [b[1]]])$;
> $L := Transpose(Matrix([|00\rangle, |01\rangle, |10\rangle, |11\rangle]))$:
> $C1 := TP(A, B)$;
> $|C1\rangle = Multiply(Transpose(C1), L)[1, 1]$;

$$A := \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$B := \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$C1 := \begin{bmatrix} a_0\ b_0 \\ a_0\ b_1 \\ a_1\ b_0 \\ a_1\ b_1 \end{bmatrix}$$

$$|C1\rangle = a_0\ b_0\ |00\rangle + a_0\ b_1\ |01\rangle + a_1\ b_0\ |10\rangle + a_1\ b_1\ |11\rangle \quad \textbf{(3)}$$

Problem 5a

Stage 1

> $G1 := TP(\mathbf{I2}, \mathbf{H});\ S1 := (Multiply(G1, C1));$
> $|S1\rangle = simplify(\ factor(\ combine(Multiply(Transpose(S1), L)[1, 1])));$

$$G1 := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 0 & 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$S1 := \begin{bmatrix} \frac{1}{2}\sqrt{2}\, a_0\, b_0 + \frac{1}{2}\sqrt{2}\, a_0\, b_1 \\ \frac{1}{2}\sqrt{2}\, a_0\, b_0 - \frac{1}{2}\sqrt{2}\, a_0\, b_1 \\ \frac{1}{2}\sqrt{2}\, a_1\, b_0 + \frac{1}{2}\sqrt{2}\, a_1\, b_1 \\ \frac{1}{2}\sqrt{2}\, a_1\, b_0 - \frac{1}{2}\sqrt{2}\, a_1\, b_1 \end{bmatrix}$$

$$|S1\rangle = \frac{1}{2}\sqrt{2}\,\big(a_0\, b_0\, |00\rangle + |00\rangle\, a_0\, b_1 + |01\rangle\, a_0\, b_0 - a_0\, b_1\, |01\rangle + a_1\, b_0\, |10\rangle + |10\rangle\, a_1\, b_1 + |11\rangle\, a_1\, b_0 - a_1\, b_1\, |11\rangle\big) \tag{4}$$

Stage 2

> $G2 := CNOT : S2 := simplify(Multiply(G2, S1));$
> $|S2\rangle = simplify(\ factor(\ combine(Multiply(Transpose(S2), L)[1, 1])));$

$$S2 := \begin{bmatrix} \frac{1}{2}\sqrt{2}\, a_0\, b_0 + \frac{1}{2}\sqrt{2}\, a_0\, b_1 \\ \frac{1}{2}\sqrt{2}\, a_0\, b_0 - \frac{1}{2}\sqrt{2}\, a_0\, b_1 \\ \frac{1}{2}\sqrt{2}\, a_1\, b_0 - \frac{1}{2}\sqrt{2}\, a_1\, b_1 \\ \frac{1}{2}\sqrt{2}\, a_1\, b_0 + \frac{1}{2}\sqrt{2}\, a_1\, b_1 \end{bmatrix}$$

$$|S2\rangle = \frac{1}{2}\sqrt{2}\,\big(a_0\, b_0\, |00\rangle + |00\rangle\, a_0\, b_1 + |01\rangle\, a_0\, b_0 - a_0\, b_1\, |01\rangle + a_1\, b_0\, |10\rangle - |10\rangle\, a_1\, b_1 + |11\rangle\, a_1\, b_0 + a_1\, b_1\, |11\rangle\big) \tag{5}$$

Stage 3

> $G3 := TP(\mathbf{I2}, \mathbf{H});\ S3 := simplify(Multiply(G3, S2));$
> $|S3\rangle = Multiply(Transpose(S3), L)[1, 1];$

$$G3 := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 0 & 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$S3 := \begin{bmatrix} a_0\ b_0 \\ a_0\ b_1 \\ a_1\ b_0 \\ -a_1\ b_1 \end{bmatrix}$$

$$|S3\rangle = a_0\ b_0\ |00\rangle + a_0\ b_1\ |01\rangle + a_1\ b_0\ |10\rangle - a_1\ b_1\ |11\rangle \tag{6}$$

Overall composition

G = G3G2G1
G|C1⟩ = |C2⟩

$$|C1\rangle = a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle$$
$$|C2\rangle = a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle - a_1b_1|11\rangle$$

> $G := Multiply(G3, Multiply(G2, G1));\ C2 := Multiply(G, C1);$
> $`|C1\rangle` = Multiply(Transpose(C1), L)[1, 1];$
> $|C2\rangle = Multiply(Transpose(C2), L)[1, 1];$

$$G := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$C2 := \begin{bmatrix} a_0\ b_0 \\ a_0\ b_1 \\ a_1\ b_0 \\ -a_1\ b_1 \end{bmatrix}$$

$$|C1\rangle = a_0\ b_0\ |00\rangle + a_0\ b_1\ |01\rangle + a_1\ b_0\ |10\rangle + a_1\ b_1\ |11\rangle$$
$$|C2\rangle = a_0\ b_0\ |00\rangle + a_0\ b_1\ |01\rangle + a_1\ b_0\ |10\rangle - a_1\ b_1\ |11\rangle \tag{7}$$

Problem 5b

Stage 1

```
> G1 := TP(H, H);
  S1 := simplify(Multiply(G1, C1));
  |S1⟩ = Multiply(Transpose(S1), L)[1, 1];
```

$$G1 := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$S1 := \begin{bmatrix} \frac{1}{2}\, a_0\, b_0 + \frac{1}{2}\, a_0\, b_1 + \frac{1}{2}\, a_1\, b_0 + \frac{1}{2}\, a_1\, b_1 \\ \frac{1}{2}\, a_0\, b_0 - \frac{1}{2}\, a_0\, b_1 + \frac{1}{2}\, a_1\, b_0 - \frac{1}{2}\, a_1\, b_1 \\ \frac{1}{2}\, a_0\, b_0 + \frac{1}{2}\, a_0\, b_1 - \frac{1}{2}\, a_1\, b_0 - \frac{1}{2}\, a_1\, b_1 \\ \frac{1}{2}\, a_0\, b_0 - \frac{1}{2}\, a_0\, b_1 - \frac{1}{2}\, a_1\, b_0 + \frac{1}{2}\, a_1\, b_1 \end{bmatrix}$$

$$|S1\rangle = \left(\frac{1}{2}\, a_0\, b_0 + \frac{1}{2}\, a_0\, b_1 + \frac{1}{2}\, a_1\, b_0 + \frac{1}{2}\, a_1\, b_1\right) |00\rangle + \left(\frac{1}{2}\, a_0\, b_0 - \frac{1}{2}\, a_0\, b_1 + \frac{1}{2}\, a_1\, b_0 - \frac{1}{2}\, a_1\, b_1\right) |01\rangle + \left(\frac{1}{2}\, a_0\, b_0 + \frac{1}{2}\, a_0\, b_1 - \frac{1}{2}\, a_1\, b_0 - \frac{1}{2}\, a_1\, b_1\right) |10\rangle + \left(\frac{1}{2}\, a_0\, b_0 - \frac{1}{2}\, a_0\, b_1 - \frac{1}{2}\, a_1\, b_0 + \frac{1}{2}\, a_1\, b_1\right) |11\rangle \tag{8}$$

Stage 2

> $G2 := CNOT : S2 := simplify(Multiply(G2, S1));$
> $|S2\rangle = Multiply(Transpose(S2), L)[1, 1];$

$$S2 := \begin{bmatrix} \frac{1}{2}\ a_0\ b_0 + \frac{1}{2}\ a_0\ b_1 + \frac{1}{2}\ a_1\ b_0 + \frac{1}{2}\ a_1\ b_1 \\ \frac{1}{2}\ a_0\ b_0 - \frac{1}{2}\ a_0\ b_1 + \frac{1}{2}\ a_1\ b_0 - \frac{1}{2}\ a_1\ b_1 \\ \frac{1}{2}\ a_0\ b_0 - \frac{1}{2}\ a_0\ b_1 - \frac{1}{2}\ a_1\ b_0 + \frac{1}{2}\ a_1\ b_1 \\ \frac{1}{2}\ a_0\ b_0 + \frac{1}{2}\ a_0\ b_1 - \frac{1}{2}\ a_1\ b_0 - \frac{1}{2}\ a_1\ b_1 \end{bmatrix}$$

$$|S2\rangle = \left(\frac{1}{2}\ a_0\ b_0 + \frac{1}{2}\ a_0\ b_1 + \frac{1}{2}\ a_1\ b_0 + \frac{1}{2}\ a_1\ b_1\right)\ |00\rangle + \left(\frac{1}{2}\ a_0\ b_0 - \frac{1}{2}\ a_0\ b_1 + \frac{1}{2}\ a_1\ b_0 - \frac{1}{2}\ a_1\ b_1\right)\ |01\rangle + \left(\frac{1}{2}\ a_0\ b_0 - \frac{1}{2}\ a_0\ b_1 - \frac{1}{2}\ a_1\ b_0 + \frac{1}{2}\ a_1\ b_1\right)\ |10\rangle + \left(\frac{1}{2}\ a_0\ b_0 + \frac{1}{2}\ a_0\ b_1 - \frac{1}{2}\ a_1\ b_0 - \frac{1}{2}\ a_1\ b_1\right)\ |11\rangle \quad \textbf{(9)}$$

Stage 3

> $G3 := TP(\mathbf{H}, \mathbf{H}); S3 := simplify(Multiply(G3, S2));$
> $|S3\rangle = Multiply(Transpose(S3), L)[1, 1];$

$$G3 := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$S3 := \begin{bmatrix} a_0\ b_0 \\ a_1\ b_1 \\ a_1\ b_0 \\ a_0\ b_1 \end{bmatrix}$$

$$|S3\rangle = a_0\ b_0\ |00\rangle + a_1\ b_1\ |01\rangle + a_1\ b_0\ |10\rangle + a_0\ b_1\ |11\rangle \quad \textbf{(10)}$$

Overall composition

G = G3G2G1
G|C1⟩ = |C2⟩

$$|C1\rangle = a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle$$
$$|C2\rangle = a_0b_0|00\rangle + a_0b_1|11\rangle + a_1b_0|10\rangle + a_1b_1|01\rangle$$

```
> G := Multiply(G3, Multiply(G2, G1)); C2 := Multiply(G, C1);
  |C1⟩= Multiply(Transpose(C1), L)[1, 1];
  |C2⟩= Multiply(Transpose(C2), L)[1, 1];
```

$$G := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$C2 := \begin{bmatrix} a_0\, b_0 \\ a_1\, b_1 \\ a_1\, b_0 \\ a_0\, b_1 \end{bmatrix}$$

$$|C1\rangle = a_0\, b_0\, |00\rangle + a_0\, b_1\, |01\rangle + a_1\, b_0\, |10\rangle + a_1\, b_1\, |11\rangle$$

$$|C2\rangle = a_0\, b_0\, |00\rangle + a_1\, b_1\, |01\rangle + a_1\, b_0\, |10\rangle + a_0\, b_1\, |11\rangle \qquad \textbf{(11)}$$

Problem 5c

Stage 1

> $G1 := Bell;\ S1 := Multiply(G1, C1);$
> $|S1\rangle = factor(combine(Multiply(Transpose(S1), L)[1, 1]));$

$$G1 := \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} & 0 \end{bmatrix}$$

$$S1 := \begin{bmatrix} \frac{1}{2}\sqrt{2}\, a_0\, b_0 + \frac{1}{2}\sqrt{2}\, a_1\, b_0 \\ \frac{1}{2}\sqrt{2}\, a_0\, b_1 + \frac{1}{2}\sqrt{2}\, a_1\, b_1 \\ \frac{1}{2}\sqrt{2}\, a_0\, b_1 - \frac{1}{2}\sqrt{2}\, a_1\, b_1 \\ \frac{1}{2}\sqrt{2}\, a_0\, b_0 - \frac{1}{2}\sqrt{2}\, a_1\, b_0 \end{bmatrix}$$

$$|S1\rangle = \frac{1}{2}\sqrt{2}\,\big(a_0\, b_0\, |00\rangle + |00\rangle\, a_1\, b_0 + a_0\, b_1\, |01\rangle + a_1\, b_1\, |01\rangle + |10\rangle\, a_0\, b_1 - |10\rangle\, a_1\, b_1 + |11\rangle\, a_0\, b_0 - |11\rangle\, a_1\, b_0\big) \tag{12}$$

At the end of this stage we have the Bell's state β_{00}, β_{01}, β_{10}, and β_{11}

$$S1 = a_0b_0(|00\rangle+|11\rangle)/\sqrt{2} + a_0b_1(|01\rangle+|10\rangle)/\sqrt{2} + a_1b_0(|00\rangle - |11\rangle)/\sqrt{2} + a_1b_1(|01\rangle - |10\rangle)/\sqrt{2}$$

$$S1 = a_0b_0\,\beta_{00} + a_0b_1\,\beta_{01} + a_1b_0\,\beta_{10} + a_1b_1\beta_{11}$$

Stage 2

> $G2 := TP(\mathbf{Z}, \mathbf{I2})$; $S2 := simplify(Multiply(G2, S1))$;
> $|S2\rangle = factor(combine(Multiply(Transpose(S2), L)[1, 1]))$;

$$G2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$S2 := \begin{bmatrix} \frac{1}{2} \sqrt{2}\, a_0\, b_0 + \frac{1}{2} \sqrt{2}\, a_1\, b_0 \\ \frac{1}{2} \sqrt{2}\, a_0\, b_1 + \frac{1}{2} \sqrt{2}\, a_1\, b_1 \\ -\frac{1}{2} \sqrt{2}\, a_0\, b_1 + \frac{1}{2} \sqrt{2}\, a_1\, b_1 \\ -\frac{1}{2} \sqrt{2}\, a_0\, b_0 + \frac{1}{2} \sqrt{2}\, a_1\, b_0 \end{bmatrix}$$

$$|S2\rangle = \frac{1}{2} \sqrt{2}\, \big(a_0\, b_0\, |00\rangle + |00\rangle\, a_1\, b_0 + a_0\, b_1\, |01\rangle + a_1\, b_1\, |01\rangle - |10\rangle\, a_0\, b_1 + |10\rangle\, a_1\, b_1 - |11\rangle\, a_0\, b_0 + |11\rangle\, a_1\, b_0\big) \quad \textbf{(13)}$$

Stage 3

> $G3 := IBell : S3 := simplify(Multiply(G3, S2))$;
> $|S3\rangle = Multiply(Transpose(S3), L)[1, 1]$;

$$S3 := \begin{bmatrix} a_1\, b_0 \\ a_1\, b_1 \\ a_0\, b_0 \\ a_0\, b_1 \end{bmatrix}$$

$$|S3\rangle = |00\rangle\, a_1\, b_0 + a_1\, b_1\, |01\rangle + a_0\, b_0\, |10\rangle + a_0\, b_1\, |11\rangle \quad \textbf{(14)}$$

Overall composition

G = G3G2G1
G|C1⟩ = |C2⟩

$$|C1\rangle = a_0 b_0|00\rangle + a_0 b_1|01\rangle + a_1 b_0|10\rangle + a_1 b_1|11\rangle$$
$$|C2\rangle = a_0 b_0|10\rangle + a_0 b_1|11\rangle + a_1 b_0|00\rangle + a_1 b_1|01\rangle$$

```
> G := Multiply(G3, Multiply(G2, G1));
  C2 := Multiply(G, C1);
  `|C1⟩` = Multiply(Transpose(C1), L)[1, 1];
  `|C2⟩` = Multiply(Transpose(C2), L)[1, 1];
```

$$G := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$C2 := \begin{bmatrix} a_1\ b_0 \\ a_1\ b_1 \\ a_0\ b_0 \\ a_0\ b_1 \end{bmatrix}$$

$$|C1\rangle = a_0\ b_0\ |00\rangle + a_0\ b_1\ |01\rangle + a_1\ b_0\ |10\rangle + a_1\ b_1\ |11\rangle$$

$$|C2\rangle = |00\rangle\ a_1\ b_0 + a_1\ b_1\ |01\rangle + a_0\ b_0\ |10\rangle + a_0\ b_1\ |11\rangle \quad \textbf{(15)}$$

Problem 5d

Stage 1

```
> G1 := TP(X, I2);
  S1 := simplify(Multiply(G1, C1));
  |S1⟩ = Multiply(Transpose(S1), L)[1, 1];
```

$$G1 := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$S1 := \begin{bmatrix} a_1\ b_0 \\ a_1\ b_1 \\ a_0\ b_0 \\ a_0\ b_1 \end{bmatrix}$$

$$|S1\rangle = |00\rangle\ a_1\ b_0 + a_1\ b_1\ |01\rangle + a_0\ b_0\ |10\rangle + a_0\ b_1\ |11\rangle \tag{16}$$

Stage 2

```
> G2 := CNOT;
  S2 := simplify(Multiply(G2, S1));
  |S2⟩ = Multiply(Transpose(S2), L)[1, 1];
```

$$G2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$S2 := \begin{bmatrix} a_1\ b_0 \\ a_1\ b_1 \\ a_0\ b_1 \\ a_0\ b_0 \end{bmatrix}$$

$$|S2\rangle = |00\rangle\ a_1\ b_0 + a_1\ b_1\ |01\rangle + a_0\ b_1\ |10\rangle + a_0\ b_0\ |11\rangle \tag{17}$$

Stage 3

```
> G3 := TP(X, I2);
  S3 := simplify(Multiply(G3, S2));
  |S3⟩= Multiply(Transpose(S3), L)[1, 1];
```

$$G3 := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$S3 := \begin{bmatrix} a_0\, b_1 \\ a_0\, b_0 \\ a_1\, b_0 \\ a_1\, b_1 \end{bmatrix}$$

$$|S3\rangle = a_0\ b_1\ \ |00\rangle + a_0\ b_0\ \ |01\rangle + a_1\ b_0\ \ |10\rangle + a_1\ b_1\ \ |11\rangle \tag{18}$$

Overall composition

G = G3G2G1
G|C1⟩ = |C2⟩

$$|C1\rangle = a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle$$
$$|C2\rangle = a_0b_0|01\rangle + a_0b_1|00\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle$$

```
> G := Multiply(G3, Multiply(G2, G1));
  C2 := Multiply(G, C1);
  `|C1⟩` = Multiply(Transpose(C1), L)[1, 1];
  `|C2⟩` = Multiply(Transpose(C2), L)[1, 1];
```

$$G := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C2 := \begin{bmatrix} a_0\, b_1 \\ a_0\, b_0 \\ a_1\, b_0 \\ a_1\, b_1 \end{bmatrix}$$

$$|C1\rangle = a_0\ b_0\ \ |00\rangle + a_0\ b_1\ \ |01\rangle + a_1\ b_0\ \ |10\rangle + a_1\ b_1\ \ |11\rangle$$
$$|C2\rangle = a_0\ b_1\ \ |00\rangle + a_0\ b_0\ \ |01\rangle + a_1\ b_0\ \ |10\rangle + a_1\ b_1\ \ |11\rangle \tag{19}$$

Bell Gate Summary

1. Bell Gate : CNOT (**H**⊗**I2**)

```
> 'C1'= C1;
S1 := simplify(Multiply(Bell, C1));
|S1⟩= factor( combine(Multiply(Transpose(S1), L)[1, 1]));
```

$$C1 = \begin{bmatrix} a_0\ b_0 \\ a_0\ b_1 \\ a_1\ b_0 \\ a_1\ b_1 \end{bmatrix}$$

$$S1 := \begin{bmatrix} \frac{1}{2}\sqrt{2}\ a_0\ b_0 + \frac{1}{2}\sqrt{2}\ a_1\ b_0 \\ \frac{1}{2}\sqrt{2}\ a_0\ b_1 + \frac{1}{2}\sqrt{2}\ a_1\ b_1 \\ \frac{1}{2}\sqrt{2}\ a_0\ b_1 - \frac{1}{2}\sqrt{2}\ a_1\ b_1 \\ \frac{1}{2}\sqrt{2}\ a_0\ b_0 - \frac{1}{2}\sqrt{2}\ a_1\ b_0 \end{bmatrix}$$

$$|S1\rangle = \frac{1}{2}\sqrt{2}\ (a_0\ b_0\ |00\rangle + |00\rangle\ a_1\ b_0 + a_0\ b_1\ |01\rangle + |01\rangle\ a_1\ b_1 + |10\rangle\ a_0\ b_1 - |10\rangle\ a_1\ b_1 + |11\rangle\ a_0\ b_0 - |11\rangle\ a_1\ b_0) \quad \textbf{(20)}$$

At the end of this stage we have the Bell's states β_{00}, β_{01}, β_{10}, and β_{11}

$$\mathbf{S1} = a_0b_0(|00\rangle+|11\rangle)/\sqrt{2} + a_0b_1(|01\rangle+|10\rangle)/\sqrt{2} + a_1b_0(|00\rangle - |11\rangle)/\sqrt{2} + a_1b_1(|01\rangle - |10\rangle)/\sqrt{2}$$

$$\mathbf{S1} = 1/\sqrt{2}\ \Sigma \sum a_ib_j(\ |0(0\oplus j)\rangle + (-1)^i\ |1(1\oplus j)\rangle\ )$$

$$\mathbf{S1} = a_0b_0\ \beta_{00} + a_0b_1\ \beta_{01} + a_1b_0\ \beta_{10} + a_1b_1\beta_{11}$$

2. Inverse Bell Gate: ($\mathbf{H} \otimes \mathbf{I2}$)CNOT

```
> 'S1'= S1;
  |S1⟩= factor( combine(Multiply(Transpose(S1), L)[1, 1]));
  S2 := simplify(Multiply(IBell, S1));
  |S2⟩= Multiply(Transpose(S2), L)[1, 1];
```

$$S1 = \begin{bmatrix} \frac{1}{2}\sqrt{2}\, a_0\, b_0 + \frac{1}{2}\sqrt{2}\, a_1\, b_0 \\ \frac{1}{2}\sqrt{2}\, a_0\, b_1 + \frac{1}{2}\sqrt{2}\, a_1\, b_1 \\ \frac{1}{2}\sqrt{2}\, a_0\, b_1 - \frac{1}{2}\sqrt{2}\, a_1\, b_1 \\ \frac{1}{2}\sqrt{2}\, a_0\, b_0 - \frac{1}{2}\sqrt{2}\, a_1\, b_0 \end{bmatrix}$$

$$|S1\rangle = \frac{1}{2}\sqrt{2}\,\left(a_0\, b_0\, |00\rangle + |00\rangle\, a_1\, b_0 + a_0\, b_1\, |01\rangle + |01\rangle\, a_1\, b_1 + |10\rangle\, a_0\, b_1 - |10\rangle\, a_1\, b_1 + |11\rangle\, a_0\, b_0 - |11\rangle\, a_1\, b_0\right)$$

$$S2 := \begin{bmatrix} a_0\, b_0 \\ a_0\, b_1 \\ a_1\, b_0 \\ a_1\, b_1 \end{bmatrix}$$

$$|S2\rangle = a_0\, b_0\, |00\rangle + a_0\, b_1\, |01\rangle + a_1\, b_0\, |10\rangle + a_1\, b_1\, |11\rangle \tag{21}$$

3. False Bell gate: CNOT (**I2**⊗**H**)

> $FBell := Multiply(CNOT,\ TP(\mathbf{I2}, \mathbf{H}));$
> $S1 := Multiply(FBell, C1);$
> $|S1\rangle = factor(\ combine(Multiply(Transpose(S1), L)[1, 1]));$

$$FBell := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$S1 := \begin{bmatrix} \frac{1}{2}\sqrt{2}\ a_0\ b_0 + \frac{1}{2}\sqrt{2}\ a_0\ b_1 \\ \frac{1}{2}\sqrt{2}\ a_0\ b_0 - \frac{1}{2}\sqrt{2}\ a_0\ b_1 \\ \frac{1}{2}\sqrt{2}\ a_1\ b_0 - \frac{1}{2}\sqrt{2}\ a_1\ b_1 \\ \frac{1}{2}\sqrt{2}\ a_1\ b_0 + \frac{1}{2}\sqrt{2}\ a_1\ b_1 \end{bmatrix}$$

$$|S1\rangle = \frac{1}{2}\sqrt{2}\ \left(a_0\ b_0\ |00\rangle + |00\rangle\ a_0\ b_1 + a_0\ b_0\ |01\rangle - a_0\ b_1\ |01\rangle + a_1\ b_0\ |10\rangle - |10\rangle\ a_1\ b_1 + |11\rangle\ a_1\ b_0 + a_1\ b_1\ |11\rangle\right) \tag{22}$$

These are not the Bell's states

$$\mathbf{S1} = a_0b_0(|00\rangle+|01\rangle)/\sqrt{2} + a_0b_1(|00\rangle - |01\rangle)/\sqrt{2} + a_1b_0(|11\rangle + |10\rangle)/\sqrt{2} + a_1b_1(|11\rangle - |10\rangle)/\sqrt{2}$$

$$\mathbf{S1} = a_0b_0|0\rangle(|0\rangle+|1\rangle)/\sqrt{2} + a_0b_1|0\rangle(|0\rangle - |1\rangle)/\sqrt{2} + a_1b_0|1\rangle(|1\rangle + |0\rangle)/\sqrt{2} + a_1b_1|1\rangle(|1\rangle - |0\rangle)/\sqrt{2}$$

$$\mathbf{S1} = \frac{1}{\sqrt{2}} \sum_{i=0}^{1} \sum_{j=0}^{1} a_i b_j(\ |i(i\oplus 0)\rangle + (-1)^j\ |i(i\oplus 1)\rangle\ )$$

4. **(I2⊗H)**Bell - Stages 1 and 2 of problem 4b

```
> IHBell := Multiply(Bell, TP(I2, H));
  S1 := simplify(Multiply(IHBell, C1));
  |S1⟩ = combine( Multiply(Transpose(S1), L)[1, 1]);
```

$$IHBell := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$S1 := \begin{bmatrix} \frac{1}{2} a_0 b_0 + \frac{1}{2} a_0 b_1 + \frac{1}{2} a_1 b_0 + \frac{1}{2} a_1 b_1 \\ \frac{1}{2} a_0 b_0 - \frac{1}{2} a_0 b_1 + \frac{1}{2} a_1 b_0 - \frac{1}{2} a_1 b_1 \\ \frac{1}{2} a_0 b_0 - \frac{1}{2} a_0 b_1 - \frac{1}{2} a_1 b_0 + \frac{1}{2} a_1 b_1 \\ \frac{1}{2} a_0 b_0 + \frac{1}{2} a_0 b_1 - \frac{1}{2} a_1 b_0 - \frac{1}{2} a_1 b_1 \end{bmatrix}$$

$$|S1\rangle = \left(\frac{1}{2} a_0 b_0 + \frac{1}{2} a_0 b_1 + \frac{1}{2} a_1 b_0 + \frac{1}{2} a_1 b_1\right) |00\rangle + \left(\frac{1}{2} a_0 b_0 - \frac{1}{2} a_0 b_1 + \frac{1}{2} a_1 b_0 - \frac{1}{2} a_1 b_1\right) |01\rangle + \left(\frac{1}{2} a_0 b_0 - \frac{1}{2} a_0 b_1 - \frac{1}{2} a_1 b_0 + \frac{1}{2} a_1 b_1\right) |10\rangle + \left(\frac{1}{2} a_0 b_0 + \frac{1}{2} a_0 b_1 - \frac{1}{2} a_1 b_0 - \frac{1}{2} a_1 b_1\right) |11\rangle \qquad \textbf{(23)}$$

$$\mathbf{S1 = a_0b_0(|00\rangle+|01\rangle+|10\rangle+|11\rangle)/2 + a_0b_1(|00\rangle - |01\rangle - |10\rangle+|11\rangle)/2}$$
$$\mathbf{+ a_1b_0(|00\rangle+|01\rangle - |10\rangle - |11\rangle)/2 + a_1b_1(|00\rangle - |01\rangle + |10\rangle - |11\rangle)/2}$$

$$\mathbf{S1 = a_0b_0(\beta_{00}+\beta_{01})/\sqrt{2} + a_0b_1(\beta_{00} - \beta_{01})/\sqrt{2} + a_1b_0(\beta_{10}+\beta_{11})/\sqrt{2} + a_1b_1(\beta_{10} - \beta_{11})/\sqrt{2}}$$