```
> restart;
```

Wigner 9j coefficients

A Maple implementation. Maple V12

These procedures are derived from the C program described by William J. Thompson.
William J. Thompson, Angular Momentum: An Illustrated Guide
to Rotational Symmetries for Physical Systems,
John Wiley and Sons Inc., 1994

The Wigner 9j symbol:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \sum_k (-1)^{2k}(2k+1) \begin{bmatrix} a & b & c \\ f & i & k \end{bmatrix} \begin{bmatrix} d & e & f \\ b & k & h \end{bmatrix} \begin{bmatrix} g & h & i \\ k & a & d \end{bmatrix}$$

elements in each row and in each column of the 9j matrix satisfy the triangle rules,

|a - b| ≤ c ≤ (a + b)
|d - e| ≤ f ≤ (d + e)
|g - h| ≤ i ≤ (g + h)
|a - d| ≤ g ≤ (a + d)
|b - e| ≤ h ≤ (b + e)
|c - f| ≤ i ≤ (c + f)

The equation is implemented in procedures E9j() and E6j() as,

$$\begin{bmatrix} j1 & j2 & j12 \\ j3 & j4 & j34 \\ j13 & j24 & J \end{bmatrix} = \sum_k (2k+1) \begin{bmatrix} j1 & J & j2 \\ j34 & k & j12 \end{bmatrix} \begin{bmatrix} j2 & j34 & j24 \\ j3 & k & j4 \end{bmatrix} \begin{bmatrix} j1 & J & j3 \\ j24 & k & j13 \end{bmatrix}$$

```
> TriangleBroken := proc( x, y, z)
        local sum;
        sum := trunc(2·( x + y + z)); # print(sum) ;
        if ((type(sum, odd)) or (y < abs(x − z)) or (y > x + z)) then
           printf("\n!! (%a,%a,%a) breaks triangle rule: try again\n", x, y, z);
           return 1;
        else return 0 end if;
     end proc:
```

```
> Δ := proc(a, b, c)
```

$$\textbf{return}\left(\sqrt{\frac{(a+b-c)!\,(a+c-b)!\,(b+c-a)!}{(a+b+c+1)!}}\right);$$

```
end proc:
```

```
> E6j := proc(a1, a2, a3, a4, a5, a6)
          local maxl, minl, n, s,
                k, kmax, kmin, p,
                t0, t1, t2, t3, t4, t5, t6;

  #  Normalization

      n := Δ(a1, a2, a5)·Δ(a1, a3, a6)·Δ(a2, a4, a6)·Δ(a3, a4, a5);  #  print(Norm = evalf(n));

  #  Minimum summation index kmin

      minl := [ 0, a1 + a4 − a5 − a6, a2 + a3 − a5 − a6];
     kmin := max(minl);             #   print(" E6j kmin ", kmin);

  #  Maximun summation index kmax

     maxl := [a1 + a2 + a3 + a4 + 1, a1 + a2 − a5, a3 + a4 − a5, a1 + a3 − a6, a2 + a4 − a6];
    kmax := min(maxl[ ]);          #      print("E6j kmax ", kmax);

  # The sum

     p := a1 + a2 + a3 + a4;
     t0 := a1 + a2 + a3 + a4 + 1;
     t1 := a5 + a6 − a1 − a4;
     t2 := a5 + a6 − a2 − a3;
     t3 := a1 + a2 − a5;
     t4 := a3 + a4 − a5;
     t5 := a1 + a3 − a6;
     t6 := a2 + a4 − a6;
```

$$s := \sum_{k=kmin}^{kmax} \frac{\left((-1)^{(p+k)}\right)\cdot(t0-k)!}{k!\,(t1+k)!\cdot(t2+k)!\cdot(t3-k)!\cdot(t4-k)!\cdot(t5-k)!\,(t6-k)!};$$

```
    return (n·s);
  end proc:
```

This procedure is a modified simplify() function

```
> SP := proc(n)
          local sign, c;
          if (evalf(n) < 0) then sign := −1 else sign := 1 end if;
          c := simplify((n·n));
          return (sign·√c);
       end proc:
```

```
> E9j := proc (x1, x2, x3,
               x4, x5, x6,
               x7, x8, x9)

            local  jlist, kmin, kmax, k, s;

     # Minimum summation index kmin
            jlist := [abs(x1 − x9), abs(x8 − x4), abs(x2 − x6) ];
            kmin := max( jlist);      #  print(  `E9j kmin = `,  kmin);

     #  Maximun summation index kmax
           jlist := [x1 + x9, x8 + x4, x2 + x6];
           kmax := min( jlist);      #    print(  `E9j kmax = `,  kmax);

    # The sum of W6j Products
           s := 0;
           for k from kmin to kmax do
               s := s + (2 k + 1) · E6j(x1, x9, x2, x6, k, x3) · E6j(x2, x6, x8, x4, k, x5) · E6j(x1, x9, x4, x8, k,
      x7);
           end do;

          return (s);
       end proc:
```

```
> W9j := proc (j1, j2, j3,
               j4, j5, j6,
               j7, j8, j9)

             local  Coeff, W;
              printf ("\nAngular Momentum: 9 j coupling coefficients \n");

             if (j1 < 0) then
                 printf ("\nEnd 9j coupling coefficients\n");
                 return 0;
            end if;

         # Testing for zero coefficient
            if ( TriangleBroken(j1, j2, j3)  ≠ 0 or
                 TriangleBroken(j4, j5, j6)  ≠ 0  or
                 TriangleBroken(j7, j8, j9)  ≠ 0  or
                TriangleBroken(j1, j4, j7)  ≠ 0  or
                TriangleBroken(j2, j5, j8)  ≠ 0  or
                TriangleBroken(j3, j6, j9)  ≠ 0 )  then
                printf( "\n");
            else
                Coeff := E9j(j1, j2, j3, j4, j5, j6, j7, j8, j9);
                W := Matrix( [ [j1, j2, j3], [j4, j5, j6], [j7, j8, j9] ]);
                print( '9 j  coefficient of ',  W = SP(Coeff ) );
                printf ("                ");
               printf ("                              = %6f \n\n", Coeff );
            end if;
       end proc:
```

Examples:

W9j(j1,j2,j12,j3,j4,j34,j13,j24,J)

> $W9j\left(2, 1, 2, \frac{3}{2}, \frac{3}{2}, 1, \frac{1}{2}, \frac{1}{2}, 1\right);$

```
Angular Momentum: 9 j coupling coefficients
```

$$9\ j\ coefficient\ of, \begin{bmatrix} 2 & 1 & 2 \\ \frac{3}{2} & \frac{3}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} = -\frac{1}{60}\sqrt{6}$$

```
= -0.040825
```

> $W9j\left(2, 2, 2, \frac{3}{2}, \frac{3}{2}, 1, \frac{1}{2}, \frac{1}{2}, 1\right);$

```
Angular Momentum: 9 j coupling coefficients
```

$$9\ j\ coefficient\ of, \begin{bmatrix} 2 & 2 & 2 \\ \frac{3}{2} & \frac{3}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} = \frac{1}{300}\sqrt{210}$$

```
= 0.048305
```

> $W9j\left(4, 2, 5, \frac{7}{2}, \frac{3}{2}, 4, \frac{1}{2}, \frac{1}{2}, 1\right);$

```
Angular Momentum: 9 j coupling coefficients
```

$$9\ j\ coefficient\ of, \begin{bmatrix} 4 & 2 & 5 \\ \frac{7}{2} & \frac{3}{2} & 4 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} = \frac{1}{180}\sqrt{21}$$

```
= 0.025459
```

> $W9j\left(4, 2, 2, \frac{7}{2}, \frac{3}{2}, 4, \frac{1}{2}, \frac{1}{2}, 1\right);$

```
Angular Momentum: 9 j coupling coefficients

!! (2,4,1) breaks triangle rule: try again
```

> $W9j(3, 3, 2, 2, 2, 2, 3, 3, 2);$

Angular Momentum: 9 j coupling coefficients

$$9\,j\ \textit{coefficient of,}\ \begin{bmatrix} 3 & 3 & 2 \\ 2 & 2 & 2 \\ 3 & 3 & 2 \end{bmatrix} = \frac{157}{14700}$$

= 0.010680

> $W9j(1, 2, 1, 2, 2, 2, 1, 2, 1);$

Angular Momentum: 9 j coupling coefficients

$$9\,j\ \textit{coefficient of,}\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix} = -\frac{1}{150}$$

= -0.006667

> $W9j(3, 3, 2, 3, 3, 2, 3, 3, 2);$

Angular Momentum: 9 j coupling coefficients

$$9\,j\ \textit{coefficient of,}\ \begin{bmatrix} 3 & 3 & 2 \\ 3 & 3 & 2 \\ 3 & 3 & 2 \end{bmatrix} = \frac{17}{4410}\sqrt{6}$$

= 0.009442

> $W9j(1, 1, 1, 1, 1, 2, 1, 1, 1);$

Angular Momentum: 9 j coupling coefficients

$$9\,j\ \textit{coefficient of,}\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{18}$$

= 0.055556

> $W9j(1, 1, 1, 1, 1, 1, 1, 1, 1);$

Angular Momentum: 9 j coupling coefficients

$$9\,j\ \textit{coefficient of,}\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 0$$

= 0.000000