```
> restart;
>
```

Wigner 3j coefficients

A Maple implementation. Maple V12

These procedures are derived from the C program described by William J. Thompson.
William J. Thompson, Angular Momentum: An Illustrated Guide
to Rotational Symmetries for Physical Systems,
John Wiley and Sons Inc., 1994

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix} = (-1)^{j_1 - j_2 - m_3} \cdot \mathrm{N} \cdot S$$

$$\mathrm{N} = \sqrt{\frac{(j3 + j1 - j2)!\,(j3 - j1 + j2)!\,(j1 + j2 - j3)!\,(j3 - m3)!\,(j3 + m3)!}{(j1 + j2 + j3 + 1)!\,(j1 - m1)!\,(j1 + m1)!\,(j2 - m2)!\,(j2 + m2)!}}$$

$$\mathrm{S} = \sum_{kmin}^{kmax} \frac{(-1)^{(kmin + j2 + m2)}(j2 + j3 + m1 - k)!\,(j1 - m1 + k)!}{(k)!\,(j3 - j1 + j2 - k)!\,(j3 - m3 - k)!\,(k + j1 - j2 + m3)!}$$

where
kmin = $maximun(0, j2 - j1 - m3)$, **and** kmax = $minimum(j3 - j1 + j2, j3 - m3)$

The Clebsch-Gordan Coefficients from Wigner 3j Coefficients

$$\langle j_1,j_2;m_1,m_2 | j_1,j_2;j_3,m_3 \rangle = (-1)^{j_1 - j_2 + m_3} \sqrt{2 j_3 + 1} \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{bmatrix}$$

******* Note: We are using the following convention *******

$$[\pm \mathrm{N}] \equiv \pm\sqrt{N}$$

```
> TriangleBroken := proc( j1, j2, j3)
        local sum;
        sum := 2·( j1 + j2 + j3);
        if ( (type(sum, odd) ) or ( j2 < abs( j1 - j3) ) or ( j2 > j1 + j3) ) then
          return 1;
        else return 0 end if;
      end proc:

> EvenOrOdd := proc( j, m)
        local sum;
        sum := 2·( j + m);   # print(sum);
        if type( sum, even) then return 0;
        else return 1 end if;
      end proc:
```

This procedure returns a signed squared coefficient.

$\pm a\sqrt{n} = \pm a^2 n = \pm \mathbf{N}$

```
> SQ := proc(n)
        local sign, c;
        if (evalf(n) < 0) then sign := -1 else sign := 1 end if;
        c := simplify((n·n)·sign);
        return (c);
    end proc:
```

```
> W3j := proc(j1, j2, j3, m1, m2, m3)
        local k, kmin, kmax, N, S, C, W, CG;
        printf("Angular Momentum\n");
        printf("3j - CG coefficient\n");
       if (j1 < 0) then
           printf("\nEnd 3j - CG coefficient\n");
           return 0;
      end if;
      if (m1 + m2 + m3 ≠ 0) then
          printf("\n!! m1+m2+m3 not zero; try again\n\n");
      else
          if (TriangleBroken(j1, j2, j3) ≠ 0) then
            printf("\n!! (%a,%a,%a) breaks triangle rule: try again\n\n", j1, j2, j3);
          else
              if (abs(m1) > j1 or abs(m2) > j2 or abs(m3) > j3) then
                printf("\n!! An m values is too big; try again\n\n");
              else
                  if (EvenOrOdd(j1, m1) + EvenOrOdd(j2, m2) + EvenOrOdd(j3, m3) > 0) then
                    printf("\n!! An m does not match a j; try again\n\n");
                  else
                      kmin := max(0, j2 - j1 - m3); kmax := min( j3 - j1 + j2, j3 - m3);
```

$$N := \sqrt{\frac{(j3+j1-j2)!(j3-j1+j2)!(j1+j2-j3)!(j3-m3)!(j3+m3)!}{(j1+j2+j3+1)!(j1-m1)!(j1+m1)!(j2-m2)!(j2+m2)!}};$$

$$S := \sum_{k=kmin}^{kmax} \frac{(-1)^{(k+j2+m2)}\cdot(j2+j3+m1-k)!(j1-m1+k)!}{(k)!(j3-j1+j2-k)!(j3-m3-k)!(k+j1-j2+m3)!};$$

$$C := (-1)^{j1-j2-m3}\cdot N\cdot S;$$

$$W := Matrix([[j1, j2, j3], [m1, m2, -m3]]);$$

$$CG := (-1)^{j1-j2-m3}\cdot\sqrt{2\cdot j3+1}\cdot C;$$

```
                      print('3 j Symbol',   W = [SQ(C)]);
                      printf("                                    ");
                      printf("    = %6f\n\n ", simplify(C));
                      printf("The Clebsh-Gordan coefficient <%a,%a;%a,%a|%a,%a;%a,%a> = [%a] =
   %lf \n\n", j1, j2, m1, m2, j1, j2, j3, -m3, SQ(CG), CG ) ;
                  end if;
              end if;
          end if;
      end if;
  end proc:
```

Examples:

W3j(j1,j2,j3,m1,m2,-m3) or W3j(j1,j2,J,m1,m2,-M)

> $W3j(1, 1, 2, 1, 1, -2)$;

Angular Momentum

3j - CG coefficient

$$3\,j\ Symbol, \left[\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 1 & 2 \end{array}\right] = \left[\frac{1}{5}\right]$$

= 0.447214

The Clebsh-Gordan coefficient <1,1;1,1|1,1;2,2> = [1] = 1.000000

> $W3j(1, 1, 2, 0, 1, -1)$;

Angular Momentum

3j - CG coefficient

$$3\,j\ Symbol, \left[\begin{array}{ccc} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] = \left[-\frac{1}{10}\right]$$

= -0.316228

The Clebsh-Gordan coefficient <1,1;0,1|1,1;2,1> = [1/2] = 0.707107

> $W3j(1, 1, 2, 1, 0, -1)$;

Angular Momentum

3j - CG coefficient

$$3\,j\ Symbol, \left[\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 0 & 1 \end{array}\right] = \left[-\frac{1}{10}\right]$$

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The Clebsh-Gordan coefficient <1,1;1,0|1,1;2,1> = [1/2] = 0.707107

> $W3j(1, 1, 2, -1, 1, 0)$;

Angular Momentum

3j - CG coefficient

$$3\,j\ Symbol, \left[\begin{array}{ccc} 1 & 1 & 2 \\ -1 & 1 & 0 \end{array}\right] = \left[\frac{1}{30}\right]$$

= 0.182574

The Clebsh-Gordan coefficient <1,1;-1,1|1,1;2,0> = [1/6] = 0.408248

> $W3j(1, 1, 2, 0, 0, 0);$

Angular Momentum
3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \left[\frac{2}{15} \right]$$

= 0.365148

The Clebsh-Gordan coefficient <1,1;0,0|1,1;2,0> = [2/3] = 0.816497

> $W3j(1, 1, 2, 1, -1, 0);$

Angular Momentum
3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \left[\frac{1}{30} \right]$$

= 0.182574

The Clebsh-Gordan coefficient <1,1;1,-1|1,1;2,0> = [1/6] = 0.408248

> $W3j(1, 1, 2, -1, 0, 1);$

Angular Momentum
3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} = \left[-\frac{1}{10} \right]$$

= -0.316228

The Clebsh-Gordan coefficient <1,1;-1,0|1,1;2,-1> = [1/2] = 0.707107

> $W3j(1, 1, 2, 0, -1, 1);$

Angular Momentum
3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix} = \left[-\frac{1}{10} \right]$$

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The Clebsh-Gordan coefficient <1,1;0,-1|1,1;2,-1> = [1/2] = 0.707107

```
> W3j(1, 1, 2,-1,-1, 2);
```

Angular Momentum

3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & -2 \end{bmatrix} = \left[\frac{1}{5} \right]$$

= 0.447214

The Clebsh-Gordan coefficient <1,1;-1,-1|1,1;2,-2> = [1] = 1.000000

```
> W3j(1, 1, 1, 0, 1,-1);
```

Angular Momentum

3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \left[\frac{1}{6} \right]$$

= 0.408248

The Clebsh-Gordan coefficient <1,1;0,1|1,1;1,1> = [-1/2] = -0.707107

```
> W3j(1, 1, 1, 1, 0,-1);
```

Angular Momentum

3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \left[-\frac{1}{6} \right]$$

= -0.408248

The Clebsh-Gordan coefficient <1,1;1,0|1,1;1,1> = [1/2] = 0.707107

```
> W3j(1, 1, 1, 0, 0, 0);
```

Angular Momentum

3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = [0]$$

= 0.000000

The Clebsh-Gordan coefficient <1,1;0,0|1,1;1,0> = [0] = 0.000000

> $W3j(1, 1, 1, 1, -1, 0);$

```
Angular Momentum
3j - CG coefficient
```

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \left[\frac{1}{6} \right]$$

```
= 0.408248

 The Clebsh-Gordan coefficient <1,1;1,-1|1,1;1,0> = [1/2] = 0.707107
```

> $W3j(1, 1, 1, -1, 1, 0);$

```
Angular Momentum
3j - CG coefficient
```

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \left[-\frac{1}{6} \right]$$

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= -0.408248

 The Clebsh-Gordan coefficient <1,1;-1,1|1,1;1,0> = [-1/2] = -0.707107
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> $W3j(1, 1, 1, 0, -1, 1);$

```
Angular Momentum
3j - CG coefficient
```

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \left[-\frac{1}{6} \right]$$

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= -0.408248

 The Clebsh-Gordan coefficient <1,1;0,-1|1,1;1,-1> = [1/2] = 0.707107
```

> $W3j(1, 1, 1, -1, 0, 1);$

```
Angular Momentum
3j - CG coefficient
```

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} = \left[\frac{1}{6} \right]$$

```
= 0.408248

 The Clebsh-Gordan coefficient <1,1;-1,0|1,1;1,-1> = [-1/2] = -0.707107
```

> $W3j(1, 1, 0, 1, -1, 0)$;

Angular Momentum

3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \left[\frac{1}{3} \right]$$

= 0.577350

The Clebsh-Gordan coefficient <1,1;1,-1|1,1;0,0> = [1/3] = 0.577350

> $W3j(1, 1, 0, -1, 1, 0)$;

Angular Momentum

3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \left[\frac{1}{3} \right]$$

= 0.577350

The Clebsh-Gordan coefficient <1,1;-1,1|1,1;0,0> = [1/3] = 0.577350

> $W3j(1, 1, 0, 0, 0, 0)$;

Angular Momentum

3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left[-\frac{1}{3} \right]$$

= -0.577350

The Clebsh-Gordan coefficient <1,1;0,0|1,1;0,0> = [-1/3] = -0.577350

> $W3j\left(2, \frac{3}{2}, \frac{5}{2}, -2, -\frac{1}{2}, \frac{5}{2}\right)$;

Angular Momentum

3j - CG coefficient

$$3\,j\,Symbol, \begin{bmatrix} 2 & \frac{3}{2} & \frac{5}{2} \\ -2 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix} = \left[-\frac{2}{21} \right]$$

= -0.308607

The Clebsh-Gordan coefficient <2,3/2;-2,-1/2|2,3/2;5/2,-5/2> = [-4/7] =
-0.755929

>