```
> restart;
> with(LinearAlgebra) :
>
```

Maple 12

2j's C-G Coefficients Matrix
j1 = j2 = 1

Defining the U matrix

$$U := Matrix\left(\left[[1, 0, 0, 0, 0, 0, 0, 0, 0], \left[0, \sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{2}}, 0, 0, 0, 0, 0\right], \left[0, 0, \sqrt{\frac{1}{6}}, 0, \sqrt{\frac{2}{3}}, 0, \sqrt{\frac{1}{6}}, 0, 0\right], \left[0, 0, 0, 0, 0, \sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{2}}, 0\right], [0, 0, 0, 0, 0, 0, 0, 0, 1], \left[0, \sqrt{\frac{1}{2}}, 0, -\sqrt{\frac{1}{2}}, 0, 0, 0, 0, 0\right], \left[0, 0, \sqrt{\frac{1}{2}}, 0, 0, 0, -\sqrt{\frac{1}{2}}, 0, 0\right], \left[0, 0, 0, 0, 0, \sqrt{\frac{1}{2}}, 0, -\sqrt{\frac{1}{2}}, 0\right], \left[0, 0, \sqrt{\frac{1}{3}}, 0, -\sqrt{\frac{1}{3}}, 0, \sqrt{\frac{1}{3}}, 0, 0\right] \right]\right);$$

(1)

$$U := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & \frac{1}{3}\sqrt{6} & 0 & \frac{1}{6}\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & -\frac{1}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 \end{bmatrix} \quad (1)$$

Determining the Inverse

> $\mathcal{U} := simplify(combine(MatrixInverse(U)));$

$$\mathcal{U} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{3}\sqrt{3} \\ 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}\sqrt{3} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{3}\sqrt{3} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2)$$

Showing that $\mathbf{U^{-1} = U^{T}}$

> $\mathcal{U} := simplify(combine(MatrixInverse(U)));$
$u := simplify(combine(Transpose(U)));$
$Equal(\mathcal{U}, u);$

$$\mathcal{U} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{3}\sqrt{3} \\ 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}\sqrt{3} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{3}\sqrt{3} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{3}\sqrt{3} \\ 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}\sqrt{3} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{3}\sqrt{3} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$true \tag{3}$$

Showing that U^{-1} U = I

> $simplify(Multiply(\mathcal{U}, U));$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Showing that $U^{\dagger} = U^{-1}$

> $\mathcal{U} := simplify(combine(MatrixInverse(U)));$
> $\mathbb{U} := simplify(combine(HermitianTranspose(U)));$
> $Equal(\mathbb{U}, \mathcal{U});$

$$\mathcal{U} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{3}\sqrt{3} \\ 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}\sqrt{3} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{3}\sqrt{3} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
U := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6}\sqrt{6} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & \frac{1}{3}\sqrt{3} \\
0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}\sqrt{3} \\
0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 \\
0 & 0 & \frac{1}{6}\sqrt{6} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{3}\sqrt{3} \\
0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

true **(5)**

State vectors:

V2 is a N by 1 matrix consisting of state vectors |j1,j2;J,M⟩ or |J,M⟩
V1 is a N by 1 matrix consisting of state vectors |j1,j2;m1,m2⟩ or |m1,m2⟩

$$\mathbf{V2 = U \cdot V1}$$
$$\mathbf{U^{-1} \cdot V2 = V1}$$

Let j1 = j2 = 1

```
> V2 := Matrix([[|2,2⟩], [|2,1⟩], [|2,0⟩], [|2,-1⟩], [|2,-2⟩],
        [|1,1⟩], [|1,0⟩], [|1,-1⟩], [|0,0⟩]]);
```

$$
V2 := \begin{bmatrix} |2,2\rangle \\ |2,1\rangle \\ |2,0\rangle \\ |2,-1\rangle \\ |2,-2\rangle \\ |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \\ |0,0\rangle \end{bmatrix} \quad \mathbf{(6)}
$$

> $V1 := Matrix([[|1,1\rangle], [|1,0\rangle], [|1,-1\rangle], [|0,1\rangle], [|0,0\rangle], [|0,-1\rangle], [|-1,1\rangle], [|-1,0\rangle], [|-1,-1\rangle]]);$

$$V1 := \begin{bmatrix} |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \\ |0,1\rangle \\ |0,0\rangle \\ |0,-1\rangle \\ |-1,1\rangle \\ |-1,0\rangle \\ |-1,-1\rangle \end{bmatrix} \tag{7}$$

V2 = U · V1

```
> M := Multiply(U, V1) :
for i from 1 to RowDimension(U) do
  print(V2[i, 1] = M[i, 1, ]);
end do;
```

$$|2,2\rangle = |1,1\rangle$$

$$|2,1\rangle = \frac{1}{2}\sqrt{2}\,|1,0\rangle + \frac{1}{2}\sqrt{2}\,|0,1\rangle$$

$$|2,0\rangle = \frac{1}{6}\sqrt{6}\,|1,-1\rangle + \frac{1}{3}\sqrt{6}\,|0,0\rangle + \frac{1}{6}\sqrt{6}\,|-1,1\rangle$$

$$|2,-1\rangle = \frac{1}{2}\sqrt{2}\,|0,-1\rangle + \frac{1}{2}\sqrt{2}\,|-1,0\rangle$$

$$|2,-2\rangle = |-1,-1\rangle$$

$$|1,1\rangle = \frac{1}{2}\sqrt{2}\,|1,0\rangle - \frac{1}{2}\sqrt{2}\,|0,1\rangle$$

$$|1,0\rangle = \frac{1}{2}\sqrt{2}\,|1,-1\rangle - \frac{1}{2}\sqrt{2}\,|-1,1\rangle$$

$$|1,-1\rangle = \frac{1}{2}\sqrt{2}\,|0,-1\rangle - \frac{1}{2}\sqrt{2}\,|-1,0\rangle$$

$$|0,0\rangle = \frac{1}{3}\sqrt{3}\,|1,-1\rangle - \frac{1}{3}\sqrt{3}\,|0,0\rangle + \frac{1}{3}\sqrt{3}\,|-1,1\rangle \tag{8}$$

V1 = U^{-1}· V2

```
> M := Multiply(𝒰, V2) :
for i from 1 to RowDimension(𝒰) do
    print(V1[i, 1] = M[i, 1]);
end do;
```

$$|1,1\rangle = |2,2\rangle$$

$$|1,0\rangle = \frac{1}{2}\sqrt{2}\ |2,1\rangle + \frac{1}{2}\sqrt{2}\ |1,1\rangle$$

$$|1,-1\rangle = \frac{1}{6}\sqrt{6}\ |2,0\rangle + \frac{1}{2}\sqrt{2}\ |1,0\rangle + \frac{1}{3}\sqrt{3}\ |0,0\rangle$$

$$|0,1\rangle = \frac{1}{2}\sqrt{2}\ |2,1\rangle - \frac{1}{2}\sqrt{2}\ |1,1\rangle$$

$$|0,0\rangle = \frac{1}{3}\sqrt{6}\ |2,0\rangle - \frac{1}{3}\sqrt{3}\ |0,0\rangle$$

$$|0,-1\rangle = \frac{1}{2}\sqrt{2}\ |2,-1\rangle + \frac{1}{2}\sqrt{2}\ |1,-1\rangle$$

$$|-1,1\rangle = \frac{1}{6}\sqrt{6}\ |2,0\rangle - \frac{1}{2}\sqrt{2}\ |1,0\rangle + \frac{1}{3}\sqrt{3}\ |0,0\rangle$$

$$|-1,0\rangle = \frac{1}{2}\sqrt{2}\ |2,-1\rangle - \frac{1}{2}\sqrt{2}\ |1,-1\rangle$$

$$|-1,-1\rangle = |2,-2\rangle \tag{9}$$

```
>
```