```
> restart;
> with(LinearAlgebra) :
```

**Maple 12**

# 3j's C-G Coefficient Matrix

# $j1 = j2 = j3 = \frac{1}{2}$

**Defining the U matrix**

$$> U := Matrix\left(\left[[1,0,0,0,0,0,0,0],\left[0,\sqrt{\frac{1}{3}},\sqrt{\frac{1}{3}},0,\sqrt{\frac{1}{3}},0,0,0\right],\left[0,0,0,\sqrt{\frac{1}{3}},0,\sqrt{\frac{1}{3}},\sqrt{\frac{1}{3}},0\right],[0,0,0,0,0,0,0,1],\left[0,\sqrt{\frac{4}{6}},-\sqrt{\frac{1}{6}},0,-\sqrt{\frac{1}{6}},0,0,0\right],\left[0,0,0,\sqrt{\frac{1}{6}},0,\sqrt{\frac{1}{6}},-\sqrt{\frac{4}{6}},0\right],\left[0,0,\sqrt{\frac{1}{2}},0,-\sqrt{\frac{1}{2}},0,0,0\right],\left[0,0,0,\sqrt{\frac{1}{2}},0,-\sqrt{\frac{1}{2}},0,0\right]\right]\right);$$

$$U := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{3}\sqrt{6} & -\frac{1}{6}\sqrt{6} & 0 & -\frac{1}{6}\sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & \frac{1}{6}\sqrt{6} & -\frac{1}{3}\sqrt{6} & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 \end{bmatrix} \quad (1)$$

**Showing that $U^{-1} = U^T$**

```
> 𝕌:= combine(MatrixInverse(U));
  u := combine(Transpose(U));
  Equal(𝕌, u);
```

$$\mathbb{U} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{3}\sqrt{6} & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{6}\sqrt{6} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{6}\sqrt{6} & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{3}\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{3}\sqrt{6} & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{6}\sqrt{6} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{6}\sqrt{6} & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{3}\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathit{true} \qquad (2)$$

**Showing that $U^{\dagger} = U^{-1}$**

```
> 𝒰:= combine(MatrixInverse(U));
  𝒰 := combine(HermitianTranspose(U));
Equal(𝒰, 𝒰);
```

$$\mathcal{U} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{3}\sqrt{6} & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{6}\sqrt{6} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{6}\sqrt{6} & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{3}\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{U} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{3}\sqrt{6} & 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{6}\sqrt{6} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{6}\sqrt{6} & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{6}\sqrt{6} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 0 & \frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{3}\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

*true* **(3)**

**Showing that $UU^{-1} = UU^{\dagger} = I$**

> *simplify*(*Multiply*(*U*, *MatrixInverse*(*U*)));

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

> *simplify*(*Multiply*(*U*, *HermitianTranspose*(*U*)));

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

## State vectors:

**V2 is a N by 1 matrix consisting of state vectors |j1,j2,j3;j12,J,M⟩ or |j12,J,M⟩**
**V1 is a N by 1 matrix consisting of state vectors |j1,j2,j3;m1,m2,m3⟩ or |m1,m2,m3⟩**

$$V2 = U \cdot V1$$
$$U^{-1} \cdot V2 = V1$$

**Let j1 = j2 = j3 = 1/2**

> *V2* := *Matrix*([[|1,3/2,3/2⟩], [|1,3/2,1/2⟩], [|1,3/2,-1/2⟩], [|1,3/2,-3/2⟩], [|1,1/2,1/2⟩], [|1,1/2,-1/2⟩], [|0,1/2,1/2⟩], [|0,1/2,-1/2⟩]]);

$$V2 := \begin{bmatrix} |1,3/2,3/2\rangle \\ |1,3/2,1/2\rangle \\ |1,3/2,-1/2\rangle \\ |1,3/2,-3/2\rangle \\ |1,1/2,1/2\rangle \\ |1,1/2,-1/2\rangle \\ |0,1/2,1/2\rangle \\ |0,1/2,-1/2\rangle \end{bmatrix} \qquad (6)$$

> *V1* := *Matrix*([[|1/2,1/2,1/2⟩], [|1/2,1/2,-1/2⟩], [|1/2,-1/2,1/2⟩], [|1/2,-1/2,-1/2⟩], [|-1/2,1/2,1/2⟩], [|-1/2,1/2,-1/2⟩], [|-1/2,-1/2,1/2⟩], [|-1/2-,1/2,-1/2⟩]]);

$$V1 := \begin{bmatrix} |1/2,1/2,1/2\rangle \\ |1/2,1/2,-1/2\rangle \\ |1/2,-1/2,1/2\rangle \\ |1/2,-1/2,-1/2\rangle \\ |-1/2,1/2,1/2\rangle \\ |-1/2,1/2,-1/2\rangle \\ |-1/2,-1/2,1/2\rangle \\ |-1/2-,1/2,-1/2\rangle \end{bmatrix} \qquad (7)$$

**V2 = U·V1**

```
> M := Multiply(U, V1) :
  for i from 1 to RowDimension(U) do
    print(V2[i, 1] = M[i, 1, ]);
  end do;
```

$$|1,3/2,3/2\rangle = |1/2,1/2,1/2\rangle$$

$$|1,3/2,1/2\rangle = \frac{1}{3}\sqrt{3}\,|1/2,1/2,-1/2\rangle + \frac{1}{3}\sqrt{3}\,|1/2,-1/2,1/2\rangle + \frac{1}{3}\sqrt{3}\,|-1/2,1/2,1/2\rangle$$

$$|1,3/2,-1/2\rangle = \frac{1}{3}\sqrt{3}\,|1/2,-1/2,-1/2\rangle + \frac{1}{3}\sqrt{3}\,|-1/2,1/2,-1/2\rangle + \frac{1}{3}\sqrt{3}\,|-1/2,-1/2,1/2\rangle$$

$$|1,3/2,-3/2\rangle = |-1/2-,1/2,-1/2\rangle$$

$$|1,1/2,1/2\rangle = \frac{1}{3}\sqrt{6}\,|1/2,1/2,-1/2\rangle - \frac{1}{6}\sqrt{6}\,|1/2,-1/2,1/2\rangle - \frac{1}{6}\sqrt{6}\,|-1/2,1/2,1/2\rangle$$

$$|1,1/2,-1/2\rangle = \frac{1}{6}\sqrt{6}\,|1/2,-1/2,-1/2\rangle + \frac{1}{6}\sqrt{6}\,|-1/2,1/2,-1/2\rangle - \frac{1}{3}\sqrt{6}\,|-1/2,-1/2,1/2\rangle$$

$$|0,1/2,1/2\rangle = \frac{1}{2}\sqrt{2}\,|1/2,-1/2,1/2\rangle - \frac{1}{2}\sqrt{2}\,|-1/2,1/2,1/2\rangle$$

$$|0,1/2,-1/2\rangle = \frac{1}{2}\sqrt{2}\,|1/2,-1/2,-1/2\rangle - \frac{1}{2}\sqrt{2}\,|-1/2,1/2,-1/2\rangle \tag{8}$$

**V1 = U$^{-1}$·V2**

```
> M := Multiply(𝒰, V2) :
  for i from 1 to RowDimension(𝒰) do
    print(V1[i, 1] = M[i, 1]);
  end do;
```

$$|1/2,1/2,1/2\rangle = |1,3/2,3/2\rangle$$

$$|1/2,1/2,-1/2\rangle = \frac{1}{3}\sqrt{3}\,|1,3/2,1/2\rangle + \frac{1}{3}\sqrt{6}\,|1,1/2,1/2\rangle$$

$$|1/2,-1/2,1/2\rangle = \frac{1}{3}\sqrt{3}\,|1,3/2,1/2\rangle - \frac{1}{6}\sqrt{6}\,|1,1/2,1/2\rangle + \frac{1}{2}\sqrt{2}\,|0,1/2,1/2\rangle$$

$$|1/2,-1/2,-1/2\rangle = \frac{1}{3}\sqrt{3}\,|1,3/2,-1/2\rangle + \frac{1}{6}\sqrt{6}\,|1,1/2,-1/2\rangle + \frac{1}{2}\sqrt{2}\,|0,1/2,-1/2\rangle$$

$$|-1/2,1/2,1/2\rangle = \frac{1}{3}\sqrt{3}\,|1,3/2,1/2\rangle - \frac{1}{6}\sqrt{6}\,|1,1/2,1/2\rangle - \frac{1}{2}\sqrt{2}\,|0,1/2,1/2\rangle$$

$$|-1/2,1/2,-1/2\rangle = \frac{1}{3}\sqrt{3}\,|1,3/2,-1/2\rangle + \frac{1}{6}\sqrt{6}\,|1,1/2,-1/2\rangle - \frac{1}{2}\sqrt{2}\,|0,1/2,-1/2\rangle$$

$$|-1/2,-1/2,1/2\rangle = \frac{1}{3}\sqrt{3}\,|1,3/2,-1/2\rangle - \frac{1}{3}\sqrt{6}\,|1,1/2,-1/2\rangle$$

$$|-1/2-,1/2,-1/2\rangle = |1,3/2,-3/2\rangle \tag{9}$$

```
>
```