

```
> restart;
```

Associated Legendre functions and Spherical Harmonics

```
> P := \frac{(-1)^{ml}}{2^l \cdot l!} \cdot (1 - w^2)^{\frac{ml}{2}} \cdot \frac{d^{(ml+l)}}{d w^{(ml+l)}} (w^2 - 1)^l :
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```
> for l from 0 to 3 do
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```
  for ml from -l to l by 1 do
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```
    print( 'l=' l, 'ml'= ml, );
```

```
    print( 'P'= simplify( subs( w = cos( \theta ), P ), 'symbolic' ) );
```

```
    print( 'Y' = \sqrt{\frac{(2 \cdot l + 1)}{4 \cdot \pi} \cdot \frac{(l - ml)!}{(l + ml)!}} \cdot simplify( subs( w = cos( \theta ), P ), 'symbolic' ) \cdot e^{I \cdot ml \cdot \phi} );
```

```
    print( );
```

```
  end do;
```

```
end do;
```

$$l = 0, ml = 0$$

$$P = 1$$

$$Y = \frac{1}{2\sqrt{\pi}}$$

$$l = 1, ml = -1$$

$$P = \frac{1}{2} \sin(\theta)$$

$$Y = \frac{1}{4} \frac{\sqrt{6} \sin(\theta) e^{-I\phi}}{\sqrt{\pi}}$$

$$l = 1, ml = 0$$

$$P = \cos(\theta)$$

$$Y = \frac{1}{2} \frac{\sqrt{3} \cos(\theta)}{\sqrt{\pi}}$$

$$l = 1, ml = 1$$

$$P = -\sin(\theta)$$

$$Y = -\frac{1}{4} \frac{\sqrt{6} \sin(\theta) e^{I\phi}}{\sqrt{\pi}}$$

$$l=2, ml=-2$$

$$P=\frac{1}{8}\sin(\theta)^2$$

$$Y=\frac{1}{8}\frac{\sqrt{30}\sin(\theta)^2e^{-2\mathrm{i}\phi}}{\sqrt{\pi}}$$

$$l=2, ml=-1$$

$$P=\frac{1}{2}\sin(\theta)\cos(\theta)$$

$$Y=\frac{1}{4}\frac{\sqrt{30}\sin(\theta)\cos(\theta)e^{-\mathrm{i}\phi}}{\sqrt{\pi}}$$

$$l=2, ml=0$$

$$P=\frac{3}{2}\cos(\theta)^2-\frac{1}{2}$$

$$Y=\frac{1}{2}\frac{\sqrt{5}\left(\frac{3}{2}\cos(\theta)^2-\frac{1}{2}\right)}{\sqrt{\pi}}$$

$$l=2, ml=1$$

$$P=-3\sin(\theta)\cos(\theta)$$

$$Y=-\frac{1}{4}\frac{\sqrt{30}\sin(\theta)\cos(\theta)e^{\mathrm{i}\phi}}{\sqrt{\pi}}$$

$$l=2, ml=2$$

$$P=3\sin(\theta)^2$$

$$Y=\frac{1}{8}\frac{\sqrt{30}\sin(\theta)^2e^{2\mathrm{i}\phi}}{\sqrt{\pi}}$$

$$l=3, ml=-3$$

$$P=\frac{1}{48}\sin(\theta)^3$$

$$Y=\frac{1}{8}\frac{\sqrt{35}\sin(\theta)^3e^{-3\mathrm{i}\phi}}{\sqrt{\pi}}$$

$$l=3, ml=-2$$

$$P = \frac{1}{8} \sin(\theta)^2 \cos(\theta)$$

$$Y = \frac{1}{8} \frac{\sqrt{210} \sin(\theta)^2 \cos(\theta) e^{-2i\phi}}{\sqrt{\pi}}$$

$$l=3, ml=-1$$

$$P = \frac{1}{8} \sin(\theta) (5 \cos(\theta)^2 - 1)$$

$$Y = \frac{1}{8} \frac{\sqrt{21} \sin(\theta) (5 \cos(\theta)^2 - 1) e^{-i\phi}}{\sqrt{\pi}}$$

$$l=3, ml=0$$

$$P = \frac{1}{2} \cos(\theta) (5 \cos(\theta)^2 - 3)$$

$$Y = \frac{1}{4} \frac{\sqrt{7} \cos(\theta) (5 \cos(\theta)^2 - 3)}{\sqrt{\pi}}$$

$$l=3, ml=1$$

$$P = -\frac{3}{2} \sin(\theta) (5 \cos(\theta)^2 - 1)$$

$$Y = -\frac{1}{8} \frac{\sqrt{21} \sin(\theta) (5 \cos(\theta)^2 - 1) e^{i\phi}}{\sqrt{\pi}}$$

$$l=3, ml=2$$

$$P = 15 \sin(\theta)^2 \cos(\theta)$$

$$Y = \frac{1}{8} \frac{\sqrt{210} \sin(\theta)^2 \cos(\theta) e^{2i\phi}}{\sqrt{\pi}}$$

$$l=3, ml=3$$

$$P = -15 \sin(\theta)^3$$

$$Y = -\frac{1}{8} \frac{\sqrt{35} \sin(\theta)^3 e^{3i\phi}}{\sqrt{\pi}}$$