```
> restart;
> interface(warnlevel = 0) :            # Maple 12
> with(LinearAlgebra) :
> with(plots) :
```

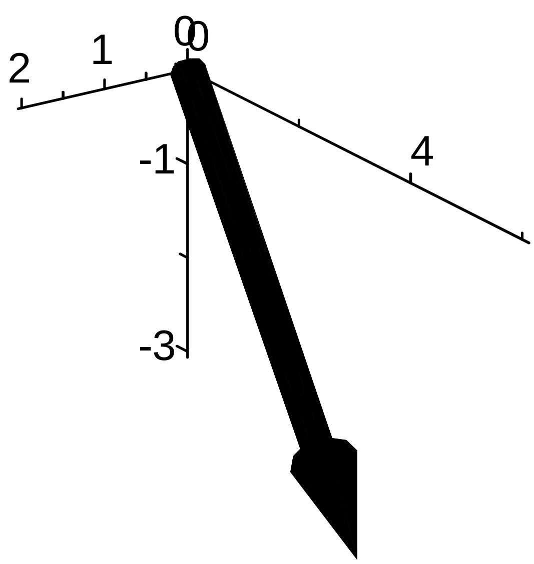
This worksheet illustrates the Givens Rotations

Let V represent a vector in 3D and r the magnitude of vector V

```
> V := Vector([[2, 6, -3]]); r := sqrt(V[1]^2 + V[2]^2 + V[3]^2);
  v1 := arrow(V,   color = black, ) :
  display([v1], axes = normal,
              scaling = constrained,  orientation = [56, 70], tickmarks = [4, 4, 4]);
```

$$V := \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$$

$$r := 7$$



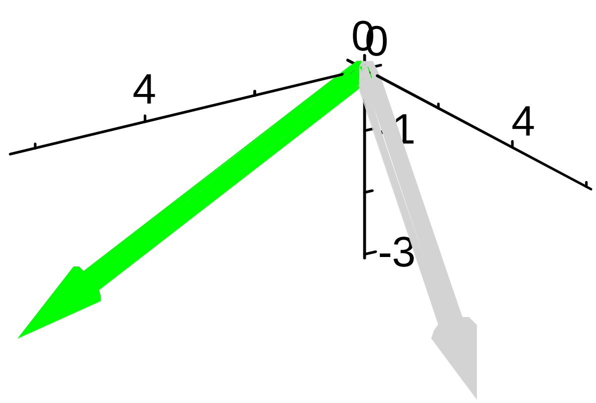
We can rotate about the z-axis and this 2D rotation takes place in the x-y plane. The y-component is zeroed-out during this process and the resulting vector lies in the x-z plane.

> $a := V[1] : b := V[2] :$
> $\mathrm{r} := \sqrt{a^2 + b^2} : \mathrm{c} := \frac{a}{r} : s := \frac{b}{r} :$
> $G1 := simplify(Matrix([[c, s, 0], [-s, c], [0, 0, 1]]));$

$$G1 := \begin{bmatrix} \frac{1}{10}\sqrt{10} & \frac{3}{10}\sqrt{10} & 0 \\ -\frac{3}{10}\sqrt{10} & \frac{1}{10}\sqrt{10} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(1)}$$

> $V3 := Multiply(G1, V);$
> $v3 := arrow(V3, color = green, axes = normal, scaling = constrained, orientation = [26, 86]) :$
> $v1 := arrow(V, color = \text{"LightGray"}, axes = normal, scaling = constrained, orientation = [56, 70], tickmarks = [4, 4, 4]) :$
> $display([v1, v3]);$

$$V3 := \begin{bmatrix} 2\sqrt{10} \\ 0 \\ -3 \end{bmatrix}$$



Notice that the rotated vector, V3 in green, now lies in the x-z plane and the magnitude has not changed

> $r3 := \sqrt{\mathrm{V3}[1]^2 + V3[2]^2 + V3[3]^2};$

$$r3 := 7 \quad \textbf{(2)}$$

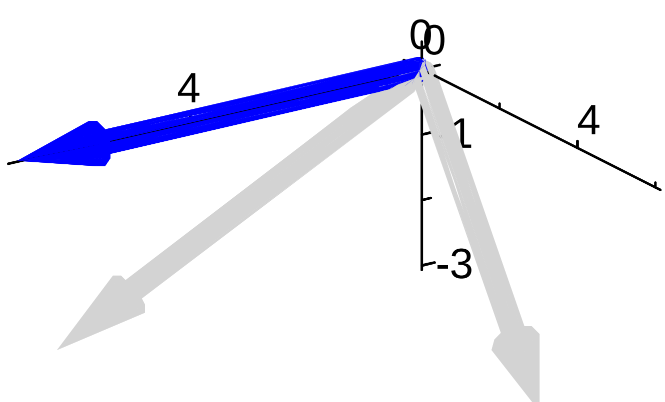
Now we rotate about the y-axis and this time the 2D rotation takes place in the x-z plane. The z-component is zeroed-out during this process and the resulting vector now lies along the x-axis

> $a := V3[1] : b := V3[3] :$
> $r := \sqrt{a^2 + b^2} : c := \frac{a}{r} : s := \frac{b}{r} :$
> $G2 := simplify(Matrix([[c, 0, s], [0, 1, 0], [-s, 0, c]]));$

$$G2 := \begin{bmatrix} \frac{2}{7}\sqrt{10} & 0 & -\frac{3}{7} \\ 0 & 1 & 0 \\ \frac{3}{7} & 0 & \frac{2}{7}\sqrt{10} \end{bmatrix} \quad (3)$$

> $V4 := Multiply(G2, V3);$
> $v4 := arrow(V4, color = blue, axes = normal,$
> $scaling = constrained, orientation = [26, 86]) :$
> $v3 := arrow(V3, color = \text{"LightGray"}, axes = normal,$
> $scaling = constrained, orientation = [26, 86]) :$
> $display([v1, v4, v3,]);$

$$V4 := \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$$



Notice the rotated vector, V4 in blue, now lies along the x-axis; no change in magnitude.

> $r4 := \sqrt{V4[1]^2 + V4[2]^2 + V4[3]^2};$

$$r4 := 7 \quad (4)$$

This procedure shows the incremental rotations.

GetRot(*V*, *r*, *zpos*, *step***)**

V **is the vector**

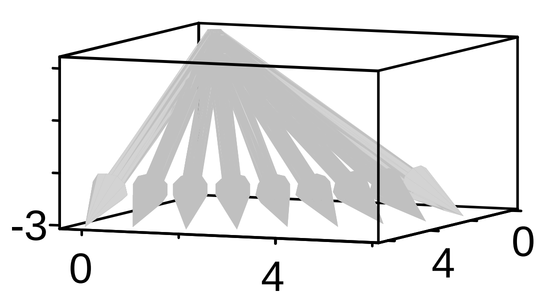
r **component of V which is not rotated to zero however, used to construct the Givens matrix.**

zpos **component of V which is rotated to zero.**

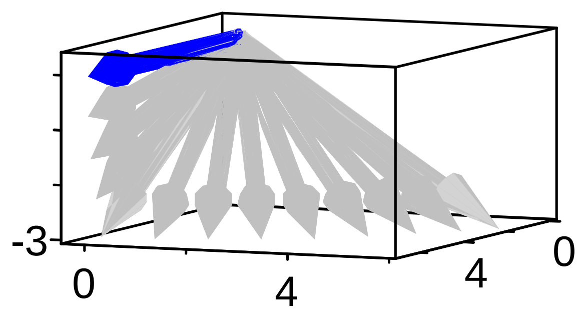
steps **increment of rotation**

```
> GetRot := proc(V, r, zpos, step)
      local i, RotM, θ, x, v, Lv, s, S;
      Lv := [ ];
      θ := evalf(arccos(V[r] / sqrt(V[r]^2 + V[zpos]^2)));
    if evalf(V[zpos]) > 0 then
        S := +1;
    else
        S := -1;
    end if;
    if zpos = 2 then
        RotM := x→Matrix([[cos(x), S · sin(x), 0], [ - S ·sin(x), cos(x), 0], [0, 0, 1]]);
    else
        RotM := x → Matrix([[cos(x), 0, S · sin(x)], [0, 1, 0], [ - S · sin(x), 0, cos(x)]]);
    end if ;
      s := θ / step;
    for i from 0 by s to θ do
        v := Multiply(RotM(i), V);
        Lv := [op(Lv), v];
     end do;
     return Lv;
  end proc:
```

```
> Lv1 := GetRot(V, 1, 2, 8) : Sv1 := arrow(Lv1, color = gray,  axes = boxed, orientation = [23, 84]) :
  display([Sv1, v1, v3]);
```



> $Lv2 := GetRot(V3, 1, 3, 4)$: $Sv2 := arrow(Lv2, color = gray, axes = boxed, orientation = [23, 84])$:
$display([Sv1, Sv2, v3, v4, v1])$;



The Givens' Rotation using Maple's built-in procedure *GivensRotationMatrix*( ).

> $V2 := Vector([[1, 2, 3, 4]])$;

$$V2 := \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \tag{5}$$

We shall zero-out the second component of V2, leaving the third and fourth components unchanged. First we obtain the Givens' Matrix, then we multiply. There is no change in magnitude.

> $G := GivensRotationMatrix(V2, 1, 2)$; $V3 := Multiply(G, V2)$;

$$G := \begin{bmatrix} \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} & 0 & 0 \\ -\frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$V3 := \begin{bmatrix} \sqrt{5} \\ 0 \\ 3 \\ 4 \end{bmatrix} \tag{6}$$

> $r2 := VectorNorm(V2, 2)$; $r3 := VectorNorm(V3, 2)$; # *Using Maple's VectorNorm() procedure*

$$r2 := \sqrt{30}$$
$$r3 := \sqrt{30} \tag{7}$$

We can also use the Given's operation to zero-out an element of a matrix. We assume that each column of the matrix represents a vector. Thus using our previous Given's matrix, we shall zero-out the second element of the first column.

> $M1 := Matrix([[1, 2, 4, 5], [2, 3, 4, 5], [3, 7, 8, 9], [4, 5, 6, 7]]);$

$$M1 := \begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 3 & 4 & 5 \\ 3 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 \end{bmatrix} \tag{8}$$

> $M2 := Multiply(G, M1);$

$$M2 := \begin{bmatrix} \sqrt{5} & \frac{8}{5}\sqrt{5} & \frac{12}{5}\sqrt{5} & 3\sqrt{5} \\ 0 & -\frac{1}{5}\sqrt{5} & -\frac{4}{5}\sqrt{5} & -\sqrt{5} \\ 3 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 \end{bmatrix} \tag{9}$$

Notice the length or magnitude of the column has not changed

> $r1 := VectorNorm(Column(M1, 1), 2); r2 := VectorNorm(\mathrm{Column}(\mathrm{M2}, 1), 2);$

$$r1 := \sqrt{30}$$
$$r2 := \sqrt{30} \tag{10}$$

We can determine the Givens matrix using our parameters, c, s, and r. Since we are zeroing-out the second component of the first column and leaving the third and fourth components unchanged, our parameters are:

> $a := V2[1] : b := V2[2] :$
> $r := \sqrt{a^2 + b^2} : \mathrm{c} := \frac{a}{r} : s := \frac{b}{r} :$

> $Gm := Matrix([[c, s, 0, 0], [-s, c, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]]);$

$$Gm := \begin{bmatrix} \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} & 0 & 0 \\ -\frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

Now let's zero-out the third component of the second column of matrix M1; we are leaving the second and fourth components unchanged.

> $G2 := GivensRotationMatrix(Column(M1, 2), 1, 3);$

$$G2 := \begin{bmatrix} \frac{2}{53}\sqrt{53} & 0 & \frac{7}{53}\sqrt{53} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{7}{53}\sqrt{53} & 0 & \frac{2}{53}\sqrt{53} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (12)$$

> $M4 := simplify(Multiply(G2, M1));$

$$M4 := \begin{bmatrix} \frac{23}{53}\sqrt{53} & \sqrt{53} & \frac{64}{53}\sqrt{53} & \frac{73}{53}\sqrt{53} \\ 2 & 3 & 4 & 5 \\ -\frac{1}{53}\sqrt{53} & 0 & -\frac{12}{53}\sqrt{53} & -\frac{17}{53}\sqrt{53} \\ 4 & 5 & 6 & 7 \end{bmatrix} \quad (13)$$

Using our parameters, c, s, and r. We are zeroing-out the third component of the second column and leaving the second and fourth components unchanged, our parameters are:

> $a := Column(M1, 2)[1] : b := Column(M1, 2)[3] : \mathrm{r} := \sqrt{a^2 + b^2} : \mathrm{c} := \frac{a}{r} : s := \frac{b}{r} :$

> $Gm := Matrix([[c, 0, s, 0], [0, 1, 0, 0], [-s, 0, c, 0], [0, 0, 0, 1]]);$

$$Gm := \begin{bmatrix} \frac{2}{53}\sqrt{53} & 0 & \frac{7}{53}\sqrt{53} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{7}{53}\sqrt{53} & 0 & \frac{2}{53}\sqrt{53} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (14)$$

> $Multiply(Gm, M1);$

$$\begin{bmatrix} \frac{23}{53}\sqrt{53} & \sqrt{53} & \frac{64}{53}\sqrt{53} & \frac{73}{53}\sqrt{53} \\ 2 & 3 & 4 & 5 \\ -\frac{1}{53}\sqrt{53} & 0 & -\frac{12}{53}\sqrt{53} & -\frac{17}{53}\sqrt{53} \\ 4 & 5 & 6 & 7 \end{bmatrix} \quad (15)$$

In our last example, we shall zero-out the fourth component of the third column, leaving the first and third components unchanged.

> $G3 := GivensRotationMatrix(Column(M1, 3), 2, 4);$

$$G3 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2}{13}\sqrt{13} & 0 & \frac{3}{13}\sqrt{13} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{3}{13}\sqrt{13} & 0 & \frac{2}{13}\sqrt{13} \end{bmatrix} \quad (16)$$

> $M5 := simplify(Multiply(G3, M1));$

$$M5 := \begin{bmatrix} 1 & 2 & 4 & 5 \\ \frac{16}{13}\sqrt{13} & \frac{21}{13}\sqrt{13} & 2\sqrt{13} & \frac{31}{13}\sqrt{13} \\ 3 & 7 & 8 & 9 \\ \frac{2}{13}\sqrt{13} & \frac{1}{13}\sqrt{13} & 0 & -\frac{1}{13}\sqrt{13} \end{bmatrix} \quad (17)$$

Using our parameters, c, s, and r. We are zeroing-out the fourth component of the third column and leaving the first and third components unchanged, our parameters are:

> $a := Column(M1, 3)[2] : b := Column(M1, 3)[4] : \mathrm{r} := \sqrt{a^2 + b^2} : \mathrm{c} := \frac{a}{r} : s := \frac{b}{r} :$

> $G4 := Matrix([[1, 0, 0, 0], [0, c, 0, s], [0, 0, 1, 0], [0, -s, 0, c]]);$

$$G4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2}{13}\sqrt{13} & 0 & \frac{3}{13}\sqrt{13} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{3}{13}\sqrt{13} & 0 & \frac{2}{13}\sqrt{13} \end{bmatrix} \quad (18)$$

> $M6 := Multiply(G4, M1);$

$$M6 := \begin{bmatrix} 1 & 2 & 4 & 5 \\ \frac{16}{13}\sqrt{13} & \frac{21}{13}\sqrt{13} & 2\sqrt{13} & \frac{31}{13}\sqrt{13} \\ 3 & 7 & 8 & 9 \\ \frac{2}{13}\sqrt{13} & \frac{1}{13}\sqrt{13} & 0 & -\frac{1}{13}\sqrt{13} \end{bmatrix} \quad (19)$$

> $VectorNorm(Column(M1, 4), 2); VectorNorm(Column(M6, 4), 2);$ # *no change in magnitude*

$$6\sqrt{5}$$

$$6\sqrt{5} \quad (20)$$