```
> restart;
> interface(warnlevel = 0) :      # Maple 12
> with(plots) :
> with(LinearAlgebra) :
```

Euler's Angles

Defining the rotation matrices using active right-handed rotation, also known as right-handed vector rotation.
Angle θ is the rotation angle about the axes.

```
> Rz := θ → Matrix([[ cos(θ),− sin(θ), 0], [ sin(θ), cos(θ), 0], [0, 0, 1]]) :
  Rx := θ → Matrix([[1, 0, 0], [0,cos(θ),− sin(θ)], [0,sin(θ), cos(θ)], ]) :
  Ry := θ → Matrix([[cos(θ), 0, sin(θ)], [0, 1, 0], [− sin(θ), 0, cos(θ)]]) :
```

These are the rotation matrices for the lab or fixed coordinate system.

Here we define a set of unit vectors, which we will use to track the object's coordinate system.

```
> Vx := Vector([1, 0, 0]);
  Vy := Vector([0, 1, 0]);
  Vz := Vector([0, 0, 1]);
```

$$Vx := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

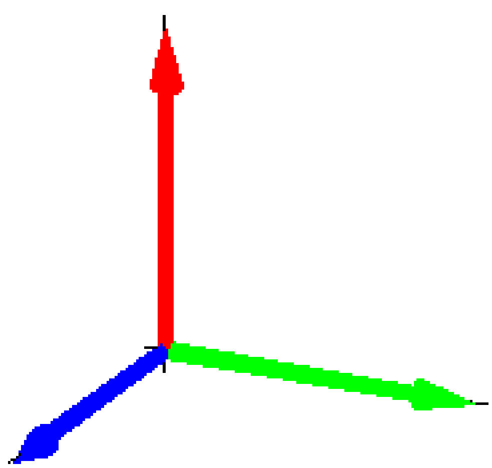
$$Vy := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Vz := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

Starting Point - Stage 0

We have two coordinate systems. The lab or fixed coordinate system and the object's coordinate system. Initially both coordinate systems coincide and we shall track the object's coordinate system by observing the rotation of unit vectors. The unit vectors have been color coded to facilitate tracking; x => blue, y => green, and z=>red. The following figure shows the initial state; the starting point.

```
> ax := arrow(Vx, color = blue) :
  ay := arrow(Vy, color = green) :
  az := arrow(Vz, color = red) :
  display([ax, ay, az], axes = normal, scaling = constrained, tickmarks = [1, 1, 1], orientation = [26, 69]);
```



Stage 1: Rz(α)

This is the first stage - we will rotate the object about the z-axis over a 45 degree angle. This is the Rz(α) rotation where alpha is 45 degrees (π/4).

Vx1 and Vy1 will be the new unit vectors and they indicate the object's x and y coordinates after the rotation.

Vx1 = Rz(α)Vx
Vy1 = Rz(α)Vy
Vz1 = Rz(α)Vz

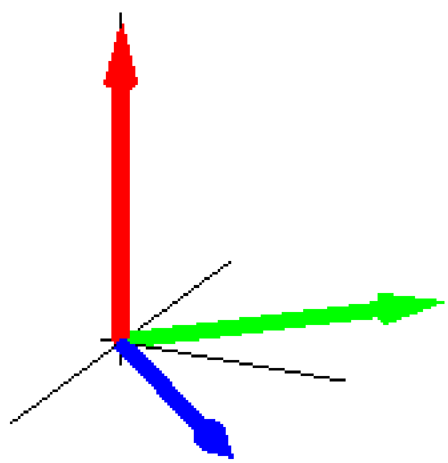
The new basis is {x', y', z}

```
> α := π/4 : # rotation angle
Vx1 := MatrixVectorMultiply(Rz(α), Vx); Vy1 := MatrixVectorMultiply(Rz(α), Vy); Vz1 :=
    MatrixVectorMultiply(Rz(α), Vz);
ax1 := arrow(Vx1, color = blue) : ay1 := arrow(Vy1, color = green) : az1 := arrow(Vz1, color = red) :
display([az1, ax1, ay1], axes = normal, tickmarks = [1, 1, 1], scaling = constrained, orientation = [26, 69]);
```

$$Vx1 := \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{bmatrix}$$

$$Vy1 := \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{bmatrix}$$

$$Vz1 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Notice that vector Vz1 is equal to Vz. This is what we expected since we are rotating about z

Stage 2: Ry'(β)

This is the second stage. Rotation about the object's y coordinate, the green unit vector, over angle β = π/4. One can easily follow this rotation, the red and blue coordinate will rotate counter-clockwise by 45 degrees. Using your right-hand point your thumb along the green vector and the red and blue vectors will rotate in the direction of the curling fingers; a right-handed rotation. Vx1 and Vy1 are related to Vx and Vy respectively by Rz(α). The rotation matrices in the {x', y', z} basis are also related to the rotation matrices in the {x, y, z} basis by the Rz(α) matrix

$$Ry'(\beta) = Rz(\alpha)Ry(\beta)Rz^{-1}(\alpha)$$
$$Ry'(\beta)Vx1 = Rz(\alpha)Ry(\beta)Rz^{-1}(\alpha)Vx1$$
$$Ry'(\beta)Vz = Rz(\alpha)Ry(\beta)Rz^{-1}(\alpha)Vz$$

We know that Vx1 = Rz(α)Vx and Vx = Rz^{-1}(α)Vx1. Also Vz1 = Vz = Rz(α)Vz and Vz = Rz^{-1}(α)Vz . Finally, let's define re-define Vz1 as Ry'(β)Vz.

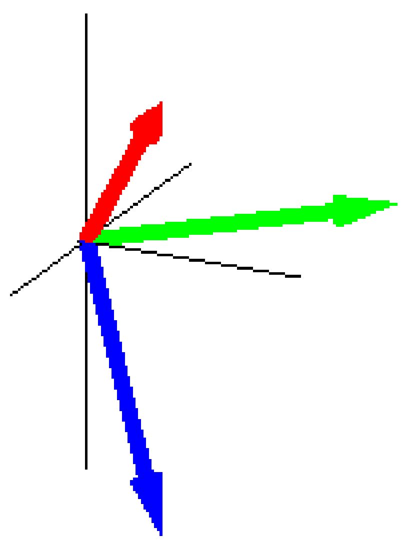
$$Vx2 = Ry'(\beta)Vx1 = Rz(\alpha)Ry(\beta)Vx$$
$$Vz1 = Ry'(\beta)Vz = Rz(\alpha)Ry(\beta)Vz$$

We now have another basis {x'', y', z'}

```
> β := π/4 : Vz1 := MatrixVectorMultiply(Rz(α), MatrixVectorMultiply(Ry(β), Vz)); az1 := arrow(Vz1, color = red) :
  Vx2 := MatrixVectorMultiply(Rz(α), MatrixVectorMultiply(Ry(β), Vx)); ax2 := arrow(Vx2, color = blue) :
  display([ax2, ay1, az1], axes = normal, tickmarks = [1, 1, 1], scaling = constrained, orientation = [26, 69]);
```

$$Vz1 := \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$Vx2 := \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2}\sqrt{2} \end{bmatrix}$$



Stage 3: Rz'(γ)

This is the third stage. Rotation about the object's z coordinate, the red unit vector, over angle $\gamma = \pi/4$. One can easily follow this rotation, the green and blue coordinate will rotate counter-clockwise by 45 degrees or using your right-hand point your thumb along the red vector and the green and blue vectors will rotate in the direction of the curling fingers; right-handed rotation. The rotation matrices in the {x'', y', z'} basis are related to the {x', y', z} basis by the Ry'(β) matrix. The new basis is {x''', y'', z'}

$$Rz'(\gamma) = Ry'(\beta)Rz(\gamma)R^{-1}y'(\beta)$$
$$Rz'(\gamma) = [\ Rz(\alpha)Ry(\beta)Rz^{-1}(\alpha)\ ]\ Rz(\gamma)R^{-1}y'(\beta)$$
$$Vx3=Rz'(\gamma)Vx2 =[\ Rz(\alpha)Ry(\beta)Rz^{-1}(\alpha)\ ]\ Rz(\gamma)R^{-1}y'(\beta)Vx2 = Rz(\alpha)Ry(\beta)Rz^{-1}(\alpha)Rz(\gamma)Vx1$$

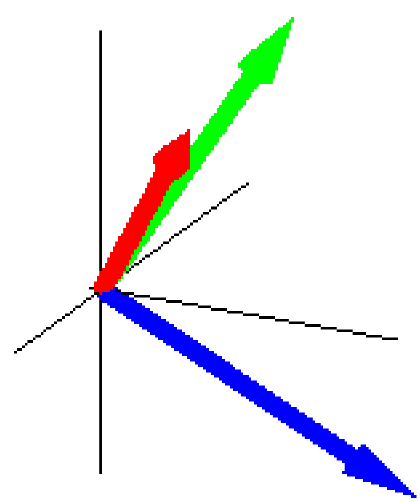
Rotations about the same axis commute: $Rz^{-1}(\alpha)Rz(\gamma) = Rz(\gamma)\,Rz^{-1}(\alpha)$

$$Vx3=Rz'(\gamma)Vx2 = Rz(\alpha)Ry(\beta)Rz(\gamma)\,Rz^{-1}(\alpha)Vx1= Rz(\alpha)Ry(\beta)Rz(\gamma)\ Vx$$
$$Vy2=Rz'(\gamma)Vy1 = Rz(\alpha)Ry(\beta)Rz(\gamma)\ Vy$$

```
> γ1 := π/4 : Vx3 := MatrixVectorMultiply(Rz(α), MatrixVectorMultiply(Ry(β), MatrixVectorMultiply(Rz(γ1), Vx)));
  Vy2 := MatrixVectorMultiply(Rz(α), MatrixVectorMultiply(Ry(β), MatrixVectorMultiply(Rz(γ1), Vy)));
  ax3 := arrow(Vx3, color = blue) : ay2 := arrow(Vy2, color = green) :
  display([ax3, ay2, az1], axes = normal, tickmarks = [1, 1, 1], scaling = constrained, orientation = [26, 69]);
```

$$Vx3 := \begin{bmatrix} \frac{1}{4}\sqrt{2} - \frac{1}{2} \\ \frac{1}{4}\sqrt{2} + \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$Vy2 := \begin{bmatrix} -\frac{1}{4}\sqrt{2} - \frac{1}{2} \\ -\frac{1}{4}\sqrt{2} + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$



The Euler Rotation matrix is given as: R(α,β,γ) = Rz(α)Ry(β)Rz(γ). Let α = β = γ1 = $\frac{\pi}{4}$

> *Erot* := *MatrixMatrixMultiply*(*Rz*(α), *MatrixMatrixMultiply*(*Ry*(β), *Rz*(γ1)));

$$Erot := \begin{bmatrix} \frac{1}{4}\sqrt{2} - \frac{1}{2} & -\frac{1}{4}\sqrt{2} - \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4}\sqrt{2} + \frac{1}{2} & -\frac{1}{4}\sqrt{2} + \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \qquad \textbf{(2)}$$

> *Vx3* := *MatrixVectorMultiply*(*Erot*, *Vx*); *Vy2* := *MatrixVectorMultiply*(*Erot*, *Vy*); *Vz1* := *MatrixVectorMultiply*(*Erot*, *Vz*);
ax3 := *arrow*(*Vx3*, *color* = *blue*) : *ay2* := *arrow*(*Vy2*, *color* = *green*) : *az1* := *arrow*(*Vz1*, *color* = *red*) :
display([*ax3*, *ay2*, *az1*], *axes* = *normal*, *tickmarks* = [1, 1, 1], *scaling* = *constrained*, *orientation* = [26, 69]);

$$Vx3 := \begin{bmatrix} \frac{1}{4}\sqrt{2} - \frac{1}{2} \\ \frac{1}{4}\sqrt{2} + \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$Vy2 := \begin{bmatrix} -\frac{1}{4}\sqrt{2} - \frac{1}{2} \\ -\frac{1}{4}\sqrt{2} + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$Vz1 := \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$$