> *restart*;

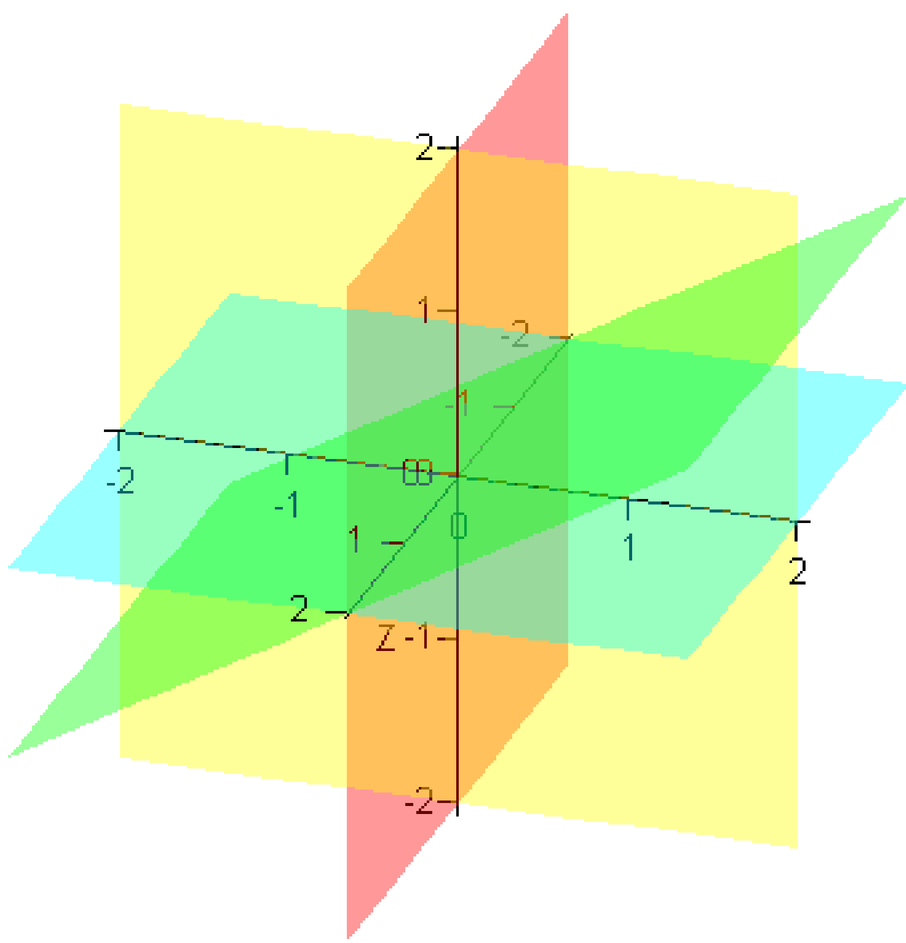
> *interface*(*warnlevel* = 0) :   #   **Maple 12**

> *with*(*LinearAlgebra*) :

> *with*(*plots*) :

3D Reflections and Rotations

> $f1 := y = 0$:   # *equation of an x – z plane in 3D space*
$f2 := x = 0$:   # *equation of an y – z plane in 3D space*
$f3 := z = 0$:   # *equation of an x – y plane in 3D space*
$f4 := -\frac{1}{2}\,y + \frac{\sqrt{3}}{2}\,z = 0$: # *equation of an x - y plane rotated 30 degrees about the x-axis*
p1 := *implicitplot3d*(*f1*, *x* = −2 ..2, *y* = 0 ..2, *z* = −2 ..2, *axes* = *normal*,
style = *patchnogrid*, *color* = *red*, *transparency* = 0.6) :
p2 := *implicitplot3d*(*f2*, *x* = 0 ..2, *y* = −2 ..2, *z* = −2 ..2, *axes* = *normal*,
style = *patchnogrid*, *color* = *yellow*, *transparency* = 0.6) :
p3 := *implicitplot3d*(*f3*, *x* = −2 ..2, *y* = −2 ..2, *z* = 0 ..2, *axes* = *normal*,
style = *patchnogrid*, *color* = *cyan*, *transparency* = 0.6) :
p4 := *implicitplot3d*(*f4*, *x* = −2 ..2, *y* = −2 ..2, *z* = −2 ..2, *axes* = *normal*,
style = *patchnogrid*, *color* = *green*, *transparency* = 0.6) :
display([*p1*, *p2*, *p3*, *p4*], axes = normal, tickmarks = [4, 4, 4], scaling = constrained, labels = [" ", " ",
Z],   *orientation* = [18, 66]) ;



3D Coordinate Rotations

> $iRxp := Matrix([[1, 0, 0], [0, \cos(x), \sin(x)], [0, -\sin(x), \cos(x),]]);$
$Rxp := Transpose(iRxp);$

$$iRxp := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \end{bmatrix}$$

$$Rxp := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(x) & -\sin(x) \\ 0 & \sin(x) & \cos(x) \end{bmatrix} \quad \textbf{(1)}$$

> $iRzp := Matrix([[\cos(x), \sin(x), 0], [-\sin(x), \cos(x), 0], [0, 0, 1]]);$
$Rzp := Transpose(iRzp);$

$$iRzp := \begin{bmatrix} \cos(x) & \sin(x) & 0 \\ -\sin(x) & \cos(x) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Rzp := \begin{bmatrix} \cos(x) & -\sin(x) & 0 \\ \sin(x) & \cos(x) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(2)}$$

> $iRyp := Matrix([[\cos(x), 0, -\sin(x)], [0, 1, 0], [\sin(x), 0, \cos(x)]]);$
$Ryp := Transpose(iRyp);$

$$iRyp := \begin{bmatrix} \cos(x) & 0 & -\sin(x) \\ 0 & 1 & 0 \\ \sin(x) & 0 & \cos(x) \end{bmatrix}$$

$$Ryp := \begin{bmatrix} \cos(x) & 0 & \sin(x) \\ 0 & 1 & 0 \\ -\sin(x) & 0 & \cos(x) \end{bmatrix} \quad \textbf{(3)}$$

3D Reflections Matrices

> $XY := Matrix([[1, 0, 0], [0, 1, 0], [0, 0, -1]]);$ # *mirror plane: x-y plane*

$$XY := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \textbf{(4)}$$

> $ZY := Matrix([[-1, 0, 0], [0, 1, 0], [0, 0, 1]]);$ # *mirror plane: z-y plane*

$$ZY := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(5)}$$

3D Reflections Matrices

> $ZX := Matrix([[1, 0, 0], [0, -1, 0], [0, 0, 1]])$; # *mirror plane: z-x plane*

$$ZX := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

> $XYx := combine(Multiply(Rxp, Multiply(XY, iRxp)))$; # *x-y plane rotated over the x-axis*

$$XYx := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(2\,x) & \sin(2\,x) \\ 0 & \sin(2\,x) & -\cos(2\,x) \end{bmatrix} \tag{7}$$

> $XYy := combine(Multiply(Ryp, Multiply(XY, iRyp)))$; #*x-y plane rotated over the y-axis*

$$XYy := \begin{bmatrix} \cos(2\,x) & 0 & -\sin(2\,x) \\ 0 & 1 & 0 \\ -\sin(2\,x) & 0 & -\cos(2\,x) \end{bmatrix} \tag{8}$$

> $ZXz := combine(Multiply(Rzp, Multiply(ZX, iRzp)))$; # *z-x plane rotated over the z-axis*

$$ZXz := \begin{bmatrix} \cos(2\,x) & \sin(2\,x) & 0 \\ \sin(2\,x) & -\cos(2\,x) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

> $ZXx := combine(Multiply(Rxp, Multiply(ZX, iRxp)))$; # *z-x plane rotated over the x-axis*

$$ZXx := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos(2\,x) & -\sin(2\,x) \\ 0 & -\sin(2\,x) & \cos(2\,x) \end{bmatrix} \tag{10}$$

> $ZYz := combine(Multiply(Rzp, Multiply(ZY, iRzp)))$; # *z-y plane rotated over the z-axis*

$$ZYz := \begin{bmatrix} -\cos(2\,x) & -\sin(2\,x) & 0 \\ -\sin(2\,x) & \cos(2\,x) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

> $ZYy := combine(Multiply(Ryp, Multiply(ZY, iRyp)))$; #*z-y plane rotated over the y-axis*

$$ZYy := \begin{bmatrix} -\cos(2\,x) & 0 & \sin(2\,x) \\ 0 & 1 & 0 \\ \sin(2\,x) & 0 & \cos(2\,x) \end{bmatrix} \tag{12}$$

Equation of a plane passing through the origin in a Cartesian coordinate system

ax + by + cz =0

$$\mathbf{N} = \frac{1}{\sqrt{a^2+b^2+c^2}}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$$

where N is a unit vector normal to the plane

The Reflection Operator: (I - 2[N*N^T])

```
> I3 := IdentityMatrix(3) :
  N := 1/sqrt((a^2+b^2+c^2)) Vector([a, b, c]);
  r := I3 - 2·Multiply(N, Transpose(N)) :
  'I - 2 N·N^T' = r;
  Hh := HouseholderMatrix(<a, b, c>); # Maple's Householder() function
```

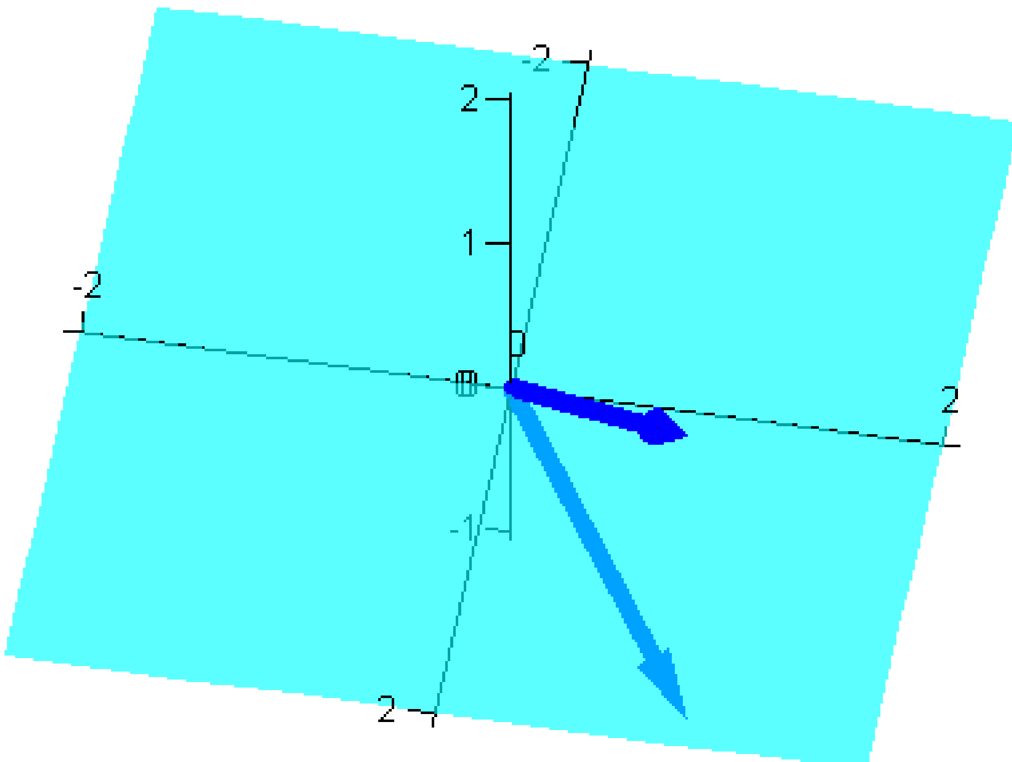
$$N := \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2+c^2}} \\ \frac{b}{\sqrt{a^2+b^2+c^2}} \\ \frac{c}{\sqrt{a^2+b^2+c^2}} \end{bmatrix}$$

$$\mathrm{I} - 2\ N\ N^T = \begin{bmatrix} 1 - \frac{2\,a^2}{a^2+b^2+c^2} & -\frac{2\,a\,b}{a^2+b^2+c^2} & -\frac{2\,a\,c}{a^2+b^2+c^2} \\ -\frac{2\,a\,b}{a^2+b^2+c^2} & 1-\frac{2\,b^2}{a^2+b^2+c^2} & -\frac{2\,b\,c}{a^2+b^2+c^2} \\ -\frac{2\,a\,c}{a^2+b^2+c^2} & -\frac{2\,b\,c}{a^2+b^2+c^2} & 1-\frac{2\,c^2}{a^2+b^2+c^2} \end{bmatrix}$$

$$Hh := \begin{bmatrix} 1 - \frac{2\,a\,\overline{a}}{a\,\overline{a}+b\,\overline{b}+c\,\overline{c}} & -\frac{2\,a\,\overline{b}}{a\,\overline{a}+b\,\overline{b}+c\,\overline{c}} & -\frac{2\,a\,\overline{c}}{a\,\overline{a}+b\,\overline{b}+c\,\overline{c}} \\ -\frac{2\,b\,\overline{a}}{a\,\overline{a}+b\,\overline{b}+c\,\overline{c}} & 1-\frac{2\,b\,\overline{b}}{a\,\overline{a}+b\,\overline{b}+c\,\overline{c}} & -\frac{2\,b\,\overline{c}}{a\,\overline{a}+b\,\overline{b}+c\,\overline{c}} \\ -\frac{2\,c\,\overline{a}}{a\,\overline{a}+b\,\overline{b}+c\,\overline{c}} & -\frac{2\,c\,\overline{b}}{a\,\overline{a}+b\,\overline{b}+c\,\overline{c}} & 1-\frac{2\,c\,\overline{c}}{a\,\overline{a}+b\,\overline{b}+c\,\overline{c}} \end{bmatrix} \quad \textbf{(13)}$$

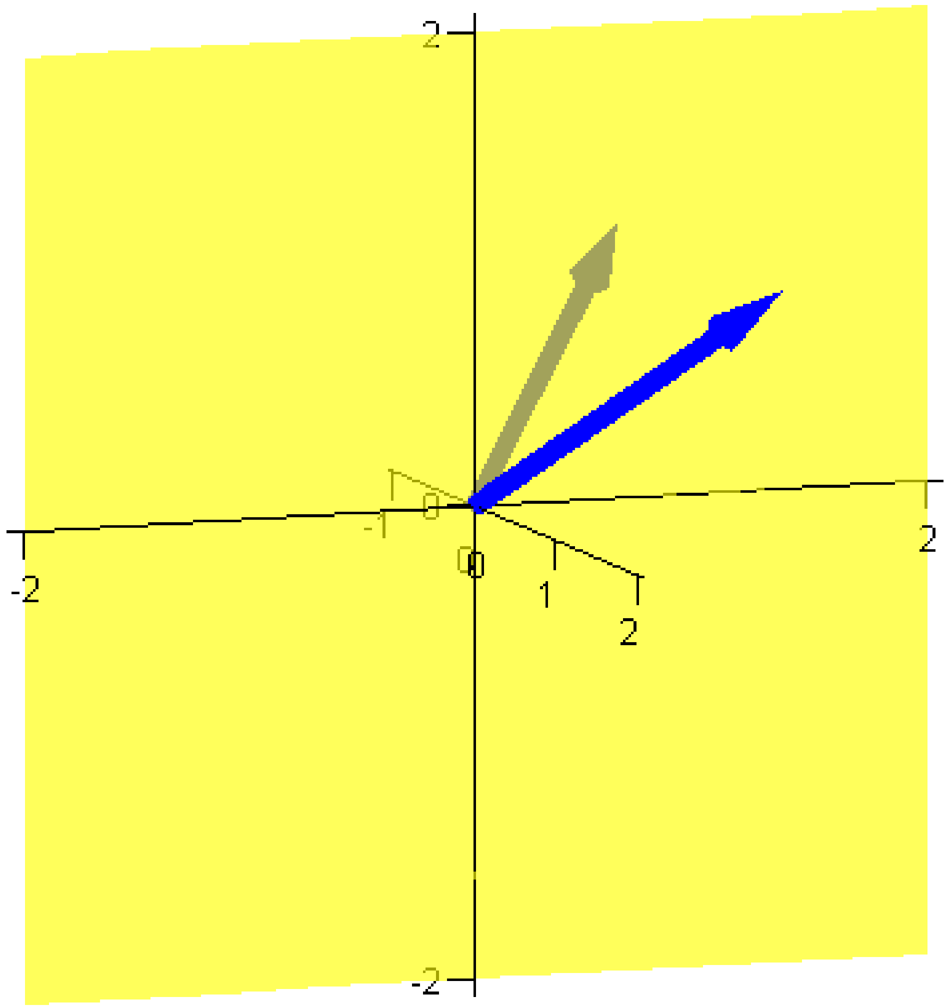
```
> x := 0 :
  v := ⟨1, 1, 1⟩ :
  P := arrow(v, color = blue) : #  vector
  V := Multiply( XYx, v) :
  Q := arrow(V, color = blue) :
  print(Reflected Vector = V);
  display( [p3, p3, P, Q],  axes = normal, scaling = constrained, orientation = [10, 41], tickmarks = [3, 3,
      3], labels = [" ", " ", " "] );
```

$$\textit{Reflected Vector} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$



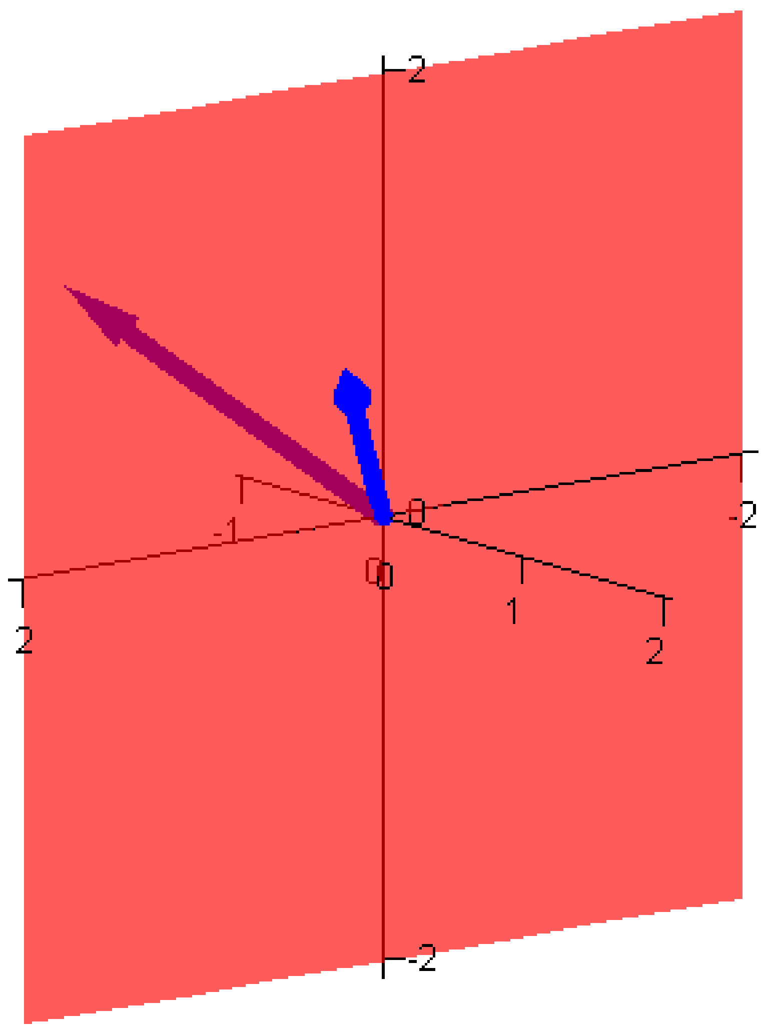
> $x := 0$:
v := ⟨1, 1, 1⟩ :
P := arrow(v, color = blue) : # *vector*
V := Multiply(ZYz, v) :
Q := arrow(V, color = blue) :
print(Reflected Vector = V);
display([p2, p2, P, Q], axes = normal, scaling = constrained, orientation = [-20, 81],
tickmarks = [3, 3, 3], *labels* = [" ", " ", " "]);

$$\mathit{Reflected\ \ Vector} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$



> $x := 0 :$
v := ⟨1, 1, 1⟩ :
P := arrow(v, color = blue) : # *vector*
V := Multiply(ZXz, v) :
Q := arrow(V, color = blue) :
print(Reflected Vector = V);
display([p1, p1, P, Q], axes = normal, scaling = constrained, orientation = [52, 77], tickmarks = [3, 3, 3], labels = [" ", " ", " "]);

$$\textit{Reflected Vector} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$



Equation of a plane passing through the origin: $-\frac{1}{2}\,y + \frac{\sqrt{3}}{2}\,z = 0 :$

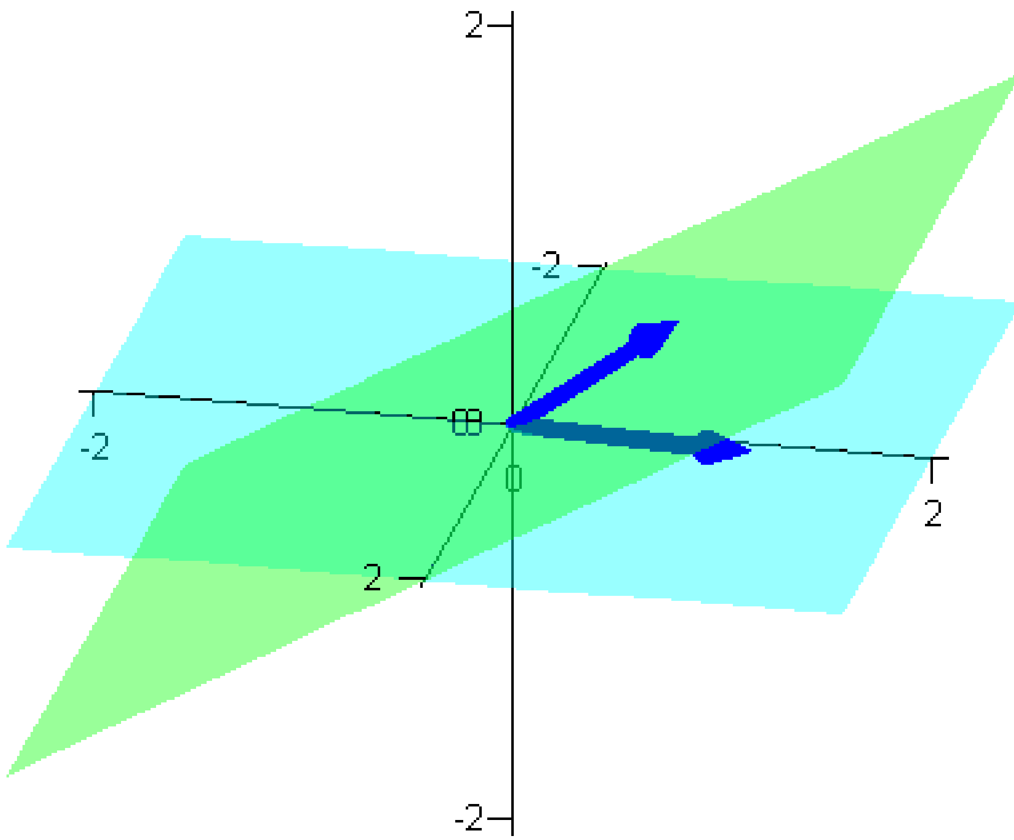
$$\mathbf{N} := \frac{1}{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

> $a := 0 : b := -\frac{1}{2} : c := \frac{\sqrt{3}}{2}$:

'$I - 2 \cdot N \cdot N^T$ ' = r; $V :=$ *Multiply*(r, ⟨1, 1, 1⟩);   # *Reflected vector*
Q := *arrow*(V, *color* = *blue*) : P := arrow(⟨1, 1, 1⟩, color = blue) : # *vector*
display([p3, p4, P, Q], axes = normal, scaling = constrained, orientation = [12, 68], *tickmarks* = [3, 3, 3], labels = [" ", " ", " "]) ;

$$I - 2\ N\ N^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ 0 & \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}$$

$$V := \begin{bmatrix} 1 \\ \frac{1}{2} + \frac{1}{2}\sqrt{3} \\ -\frac{1}{2} + \frac{1}{2}\sqrt{3} \end{bmatrix}$$



Compare with rotating the plane by 30 degrees CW, performing a reflection and rotating back.

```
> v := ⟨1, 1, 1⟩ : x := π/6 : # 30 degrees
  V := Multiply( XYx, v) : Q := arrow(V, color = blue) : P := arrow(v, color = blue) : #  vector
  'XYx' = XYx; print(Reflected Vector = V);
  display( [p3, p4, P, Q],  axes = normal, scaling = constrained, orientation = [12, 68], tickmarks = [3, 3,
     3],
                      labels = [" ", " ", " "] );
```

$$XYx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ 0 & \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}$$

$$Reflected\ \ Vector = \begin{bmatrix} 1 \\ \frac{1}{2} + \frac{1}{2}\sqrt{3} \\ -\frac{1}{2} + \frac{1}{2}\sqrt{3} \end{bmatrix}$$