```
> restart :
> interface(warnlevel = 0) :            #   Maple 12
> with(LinearAlgebra) :
```

Examples of Matrix operations

TP(M1,M2): A procedure to determine the Kronecker or Tensor Product of two matrices.

TPM = M1 ⊗ M2

where M1 is an m by n matrix, M2 is a p by q matrix and TPM is an mp by nq matrix

with TPM[ithrow..Nthrow, ithcolumn..Nthcolumn] = (M1[i,j])M2

```
> TP :=proc(M1, M2)
      local TPM, i, j, m, p, n, q, irow, Nrow, icol, Ncol;
      m := RowDimension(M1);
      n := ColumnDimension(M1);
      p := RowDimension(M2);
      q := ColumnDimension(M2);
      TPM := Matrix(m·p, n·q);
      for i to m do
          for j to n do
             irow := 1 + (i − 1) ·p;
             Nrow := p + irow − 1;
             icol := 1 + ( j − 1) ·q;
             Ncol := q + icol − 1;
             TPM[irow..Nrow, icol..Ncol] := ScalarMultiply(M2, M1[i, j])
          end do
      end do;
     return TPM
     end proc :
```

Defining Test Matrices and Vectors

```
> I2 := IdentityMatrix(2);    # generates  a  2  by  2  identity  matrix
```

$$I2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (1)$$

```
> I4 := IdentityMatrix(4);  #   generates  a  4  by  4  identity  matrix
```

$$I4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

> $AllOne := ConstantMatrix(1, 4);$ # *generates a 4 by 4 matrix of 1's*

$$AllOne := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \textbf{(3)}$$

> $\mathbb{G}23 := \mathrm{RowOperation}(\mathbb{I}4, [2, 3]);$ # *swaps rows 2 and 3 of a 4 by 4 identity matrix*

$$\mathbb{G}23 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(4)}$$

> $\mathrm{CNOT} := RowOperation(\mathbb{I}4, [3, 4]);$ # *swaps rows 3 and 4*

$$CNOT := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \textbf{(5)}$$

> $A := Matrix([[a[11], a[12], a[13]], [a[21], a[22], a[23]], [a[31], a[32], a[33]]]);$

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \textbf{(6)}$$

> $B := Matrix([[b[11], b[12]], [b[21], b[22]]]);$

$$B := \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad \textbf{(7)}$$

Vectors expressed as M by 1 matrices

> $\mathrm{V1} := Matrix([[a], [b], [c]]);$ # *vector V1 as a 3 by 1 matrix*

$$V1 := \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \textbf{(8)}$$

> $\mathrm{V2} := Matrix([[m], [n], [p]]);$ # *vector V2 as a 3 by 1 matrix*

$$V2 := \begin{bmatrix} m \\ n \\ p \end{bmatrix} \quad \textbf{(9)}$$

Multiplication of a Matrix by a scalar s using : *ScalarMultiply()*

> $V := ScalarMultiply(V1, s);$

$$V := \begin{bmatrix} s\,a \\ s\,b \\ s\,c \end{bmatrix} \tag{10}$$

Multiplication of a Matrix by a Matrix using : *MatrixMatrixMultiply() or Multiply()*

> $MatrixMatrixMultiply(\mathbb{G}23,\ CNOT);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{11}$$

> $MatrixMatrixMultiply(CNOT, \mathbb{G}23);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{12}$$

> $Multiply(CNOT, \mathbb{G}23);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{13}$$

Kronecker Product using : *TP()*

> $TP\left(\mathbb{G}23,\ \mathbb{I}2\right);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

> $TP(\mathbb{I}2, \mathbb{G}23);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

> $TP(\mathrm{V1}, \mathrm{V2});$

$$\begin{bmatrix} a\,m \\ a\,n \\ a\,p \\ b\,m \\ b\,n \\ b\,p \\ c\,m \\ c\,n \\ c\,p \end{bmatrix} \tag{16}$$

> $TP(\mathrm{V2}, \mathrm{V1});$

$$\begin{bmatrix} a\,m \\ b\,m \\ c\,m \\ a\,n \\ b\,n \\ c\,n \\ a\,p \\ b\,p \\ c\,p \end{bmatrix} \tag{17}$$

```
> Id := IdentityMatrix(3) :
  TP(A, Id);
  TP(Id, A);
```

$$\begin{bmatrix} a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} \\ a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} \\ a_{31} & 0 & 0 & a_{32} & 0 & 0 & a_{33} & 0 & 0 \\ 0 & a_{31} & 0 & 0 & a_{32} & 0 & 0 & a_{33} & 0 \\ 0 & 0 & a_{31} & 0 & 0 & a_{32} & 0 & 0 & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \textbf{(18)}$$

Kronecker Product using Maple's KroneckerProduct function: KroneckerProduct()

> $KroneckerProduct(\mathbb{G}23,\ \mathbb{I}2);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

> $KroneckerProduct(\mathbb{I}2, \mathbb{G}23);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

> $KroneckerProduct(\mathrm{V1}, \mathrm{V2});$

$$\begin{bmatrix} a\,m \\ a\,n \\ a\,p \\ b\,m \\ b\,n \\ b\,p \\ c\,m \\ c\,n \\ c\,p \end{bmatrix} \tag{21}$$

> $KroneckerProduct(\mathrm{V2},\ \mathrm{V1});$

$$\begin{bmatrix} a\,m \\ b\,m \\ c\,m \\ a\,n \\ b\,n \\ c\,n \\ a\,p \\ b\,p \\ c\,p \end{bmatrix} \quad \textbf{(22)}$$

The Trace of A⊗B = C

> $C := TP(A, B);$

$$C := \begin{bmatrix} a_{11}\,b_{11} & a_{11}\,b_{12} & a_{12}\,b_{11} & a_{12}\,b_{12} & a_{13}\,b_{11} & a_{13}\,b_{12} \\ a_{11}\,b_{21} & a_{11}\,b_{22} & a_{12}\,b_{21} & a_{12}\,b_{22} & a_{13}\,b_{21} & a_{13}\,b_{22} \\ a_{21}\,b_{11} & a_{21}\,b_{12} & a_{22}\,b_{11} & a_{22}\,b_{12} & a_{23}\,b_{11} & a_{23}\,b_{12} \\ a_{21}\,b_{21} & a_{21}\,b_{22} & a_{22}\,b_{21} & a_{22}\,b_{22} & a_{23}\,b_{21} & a_{23}\,b_{22} \\ a_{31}\,b_{11} & a_{31}\,b_{12} & a_{32}\,b_{11} & a_{32}\,b_{12} & a_{33}\,b_{11} & a_{33}\,b_{12} \\ a_{31}\,b_{21} & a_{31}\,b_{22} & a_{32}\,b_{21} & a_{32}\,b_{22} & a_{33}\,b_{21} & a_{33}\,b_{22} \end{bmatrix} \quad \textbf{(23)}$$

> $'tr(A \otimes B)' = Trace(C);$
> $'tr(A \otimes B)' = factor(Trace(C));$

$$tr(A \otimes B) = a_{11}\,b_{11} + a_{11}\,b_{22} + a_{22}\,b_{11} + a_{22}\,b_{22} + a_{33}\,b_{11} + a_{33}\,b_{22}$$

$$tr(A \otimes B) = (b_{11} + b_{22})\,(a_{11} + a_{22} + a_{33}) \quad \textbf{(24)}$$

The Trace of B⊗A

> $E := TP(B, A);$

$$E := \begin{bmatrix} a_{11}\,b_{11} & a_{12}\,b_{11} & a_{13}\,b_{11} & a_{11}\,b_{12} & a_{12}\,b_{12} & a_{13}\,b_{12} \\ a_{21}\,b_{11} & a_{22}\,b_{11} & a_{23}\,b_{11} & a_{21}\,b_{12} & a_{22}\,b_{12} & a_{23}\,b_{12} \\ a_{31}\,b_{11} & a_{32}\,b_{11} & a_{33}\,b_{11} & a_{31}\,b_{12} & a_{32}\,b_{12} & a_{33}\,b_{12} \\ a_{11}\,b_{21} & a_{12}\,b_{21} & a_{13}\,b_{21} & a_{11}\,b_{22} & a_{12}\,b_{22} & a_{13}\,b_{22} \\ a_{21}\,b_{21} & a_{22}\,b_{21} & a_{23}\,b_{21} & a_{21}\,b_{22} & a_{22}\,b_{22} & a_{23}\,b_{22} \\ a_{31}\,b_{21} & a_{32}\,b_{21} & a_{33}\,b_{21} & a_{31}\,b_{22} & a_{32}\,b_{22} & a_{33}\,b_{22} \end{bmatrix} \quad \textbf{(25)}$$

> $'tr(B \otimes A)' = factor(Trace(E));$

$$tr(B \otimes A) = (b_{11} + b_{22})\,(a_{11} + a_{22} + a_{33}) \quad \textbf{(26)}$$

The Determinant of $A\otimes B$

$\det(A\otimes B) = (\det A)^2 (\det B)^3 = \det A^2 \det B^3$

> 'det(C)'= factor(Determinant(C));

$$det(C) = (b_{22} b_{11} - b_{12} b_{21})^3 (a_{13} a_{21} a_{32} + a_{22} a_{33} a_{11} - a_{12} a_{21} a_{33} - a_{32} a_{23} a_{11} - a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31})^2 \tag{27}$$

> 'det(A)'= factor(Determinant(A));
> 'det(B)'= factor(Determinant(B));

$$det(A) = a_{13} a_{21} a_{32} + a_{22} a_{33} a_{11} - a_{12} a_{21} a_{33} - a_{32} a_{23} a_{11} - a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31}$$

$$det(B) = b_{22} b_{11} - b_{12} b_{21} \tag{28}$$

> factor(Determinant(Multiply(A, A)));
> factor(Determinant(Multiply(B, Multiply(B, B))));

$$(a_{13} a_{21} a_{32} + a_{22} a_{33} a_{11} - a_{12} a_{21} a_{33} - a_{32} a_{23} a_{11} - a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31})^2$$

$$(b_{22} b_{11} - b_{12} b_{21})^3 \tag{29}$$

The Determinant of $B\otimes A = E$

> 'det(E)'= factor(Determinant(E));

$$det(E) = (b_{22} b_{11} - b_{12} b_{21})^3 (a_{13} a_{21} a_{32} + a_{22} a_{33} a_{11} - a_{12} a_{21} a_{33} - a_{32} a_{23} a_{11} - a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31})^2 \tag{30}$$

>