```
> restart;
> with(LinearAlgebra) :                #` Maple 12
```

This is Chapter 1 Matrix Problem 4

Defining the R matrix

```
> R := Matrix([[1, 0, −1], [0, 1, 1], [−1, 1, 1]]);
```

$$R := \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \tag{1}$$

The elements of R are real, the matrix is symmetric and equals its own transpose. The eigenvalues will be real and the corresponding eigenvectors will be orthogonal. A matrix whose column consists of the eigenvectors of R is an orthogonal matrix (a unitary matrix whose elements are real) and its inverse is also its transpose. Matrix R is a Hermitian matrix.

```
> ℛ := Transpose(R);
```

$$\mathscr{R} := \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \tag{2}$$

```
> ℛ := HermitianTranspose(R);
```

$$\mathscr{R} := \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \tag{3}$$

Solving for the eigenvalues of matrix R; obtaining the roots of the characteristic polynomial

```
> CharacteristicPolynomial(R, λ);
  factor(% );
  solve( %= 0, [λ]);
```

$$1 + \lambda^3 - 3\,\lambda^2 + \lambda$$

$$(\lambda - 1)\,(\lambda^2 - 2\,\lambda - 1)$$

$$[[\lambda = 1], [\lambda = 1 + \sqrt{2}], [\lambda = 1 - \sqrt{2}]] \tag{4}$$

Determining the eigenvalues and corresponding eigenvectors of matrix R using Maple's Eigenvectors() function

> $L := Eigenvectors(R)$:   # a list of the eigenvalues with their corresponding eigenvectors
$print(eigenvalue = L[1][1],\ eigenvector = L[2][1..3, 1], magnitude = Norm(L[2][1..3, 1], Euclidean));$
$print(eigenvalue = L[1][2],\ eigenvector = L[2][1..3, 2], magnitude = Norm(L[2][1..3, 2], Euclidean));$
$print(eigenvalue = L[1][3],\ eigenvector = L[2][1..3, 3], magnitude = Norm(L[2][1..3, 3], Euclidean));$

$$eigenvalue = 1, eigenvector = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, magnitude = \sqrt{2}$$

$$eigenvalue = 1 + \sqrt{2}, eigenvector = \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 1 \end{bmatrix}, magnitude = \sqrt{2}$$

$$eigenvalue = 1 - \sqrt{2}, eigenvector = \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \\ 1 \end{bmatrix}, magnitude = \sqrt{2} \qquad (5)$$

Defining the S matrix as a matrix whose columns are the eigenvectors of matrix R

> $S := \frac{1}{\sqrt{2}} L[2];$

$$S := \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \qquad (6)$$

Matrix S is an orthogonal matrix

The column inner product of matrix S is zero; columns (eigenvectors of matrix R) are orthogonal

```
> c1 • c2 = DotProduct(S[1..3, 1], S[1..3, 2]);
  c1 • c3 = DotProduct(S[1..3, 1], S[1..3, 3]);
  c2 • c3 = DotProduct(S[1..3, 2], S[1..3, 3]);
```

$$c1 \cdot c2 = 0$$
$$c1 \cdot c3 = 0$$
$$c2 \cdot c3 = 0 \tag{7}$$

The row inner product of matrix S is also zero.

```
> r1 • r2 = DotProduct(S[1, 1..3], S[2, 1..3]);
  r1 • r3 = DotProduct(S[1, 1..3], S[3, 1..3]);
  r2 • r3 = DotProduct(S[2, 1..3], S[3, 1..3]);
```

$$r1 \cdot r2 = 0$$
$$r1 \cdot r3 = 0$$
$$r2 \cdot r3 = 0 \tag{8}$$

The length of the columns and rows of matrix S is 1

```
> print(`||c_1||` = Norm(S[1..3, 1], Euclidean));
  print(`||c_2||` = Norm(S[1..3, 2], Euclidean));
  print(`||c_3||` = Norm(S[1..3, 3], Euclidean));
  print( );
  print(`||r_1||` = Norm(S[1, 1..3], Euclidean));
  print(`||r_2||` = Norm(S[2, 1..3], Euclidean));
  print(`||r_3||` = Norm(S[3, 1..3], Euclidean));
```

$$||c1|| = 1$$
$$||c2|| = 1$$
$$||c3|| = 1$$

$$||r1|| = 1$$
$$||r2|| = 1$$
$$||r3|| = 1 \tag{9}$$

The inverse of an orthogonal matrix is its own transpose. Thus $S^{-1} = S^T$

```
> ST := Transpose(S);
```

$$ST := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \tag{10}$$

> $\mathbb{S} := MatrixInverse(S);$

$$\mathbb{S} := \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \tag{11}$$

> $Equal(ST, \mathbb{S});$

$$true \tag{12}$$

Determining Λ

$$\mathbf{\Lambda = S^{-1}\, R\, S}$$

> $\Lambda := simplify(Multiply(\mathbb{S}, Multiply(R, S)));$

$$\Lambda := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\sqrt{2} & 0 \\ 0 & 0 & 1-\sqrt{2} \end{bmatrix} \tag{13}$$