```
> restart;
> interface(warnlevel = 0) :   # ` `Maple 12
> with(plots) :
> with(LinearAlgebra) :
```

Define the basis vectors:

basis set 1 { x, y, z } ; the standard basis

basis set 2 { v1, v2, v3 }

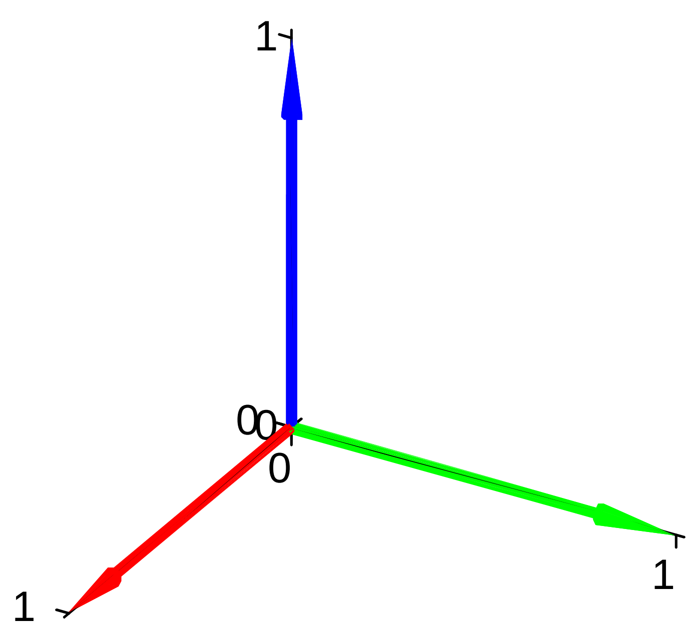
Notice that these are orthogonal bases: x • y = x · z = y · z = 0

v1 · v2 = v1 · v3 = v2 · v3 =0

```
> e1 := Vector([1, 0, 0]) :
  e2 := Vector([0, 1, 0]) :
  e3 := Vector([0, 0, 1]) :
  v1 := Vector([1, −1, 1]) :
  v2 := Vector([−1, 1, 2]) :
  v3 := Vector([1, 1, 0]) :
```

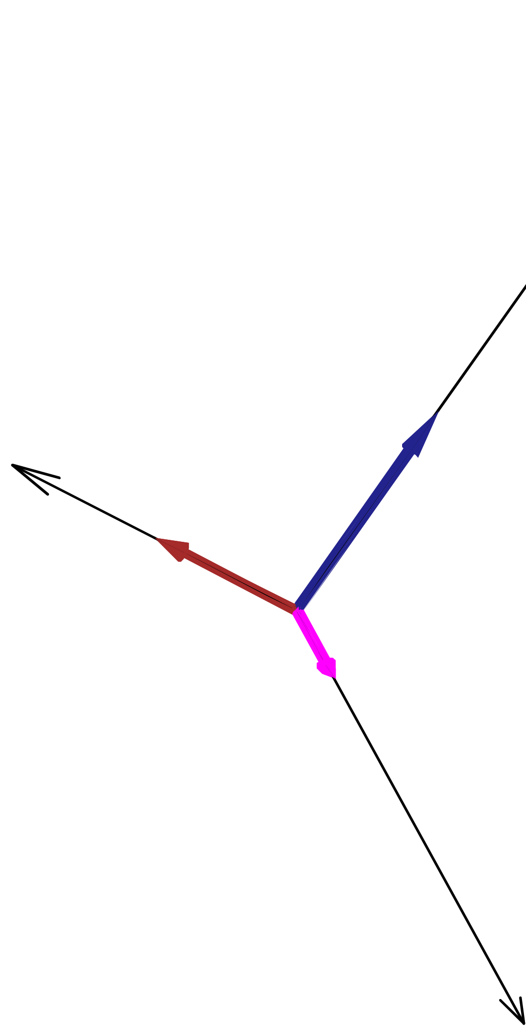
Plotting basis set 1

```
> x := arrow(e1,  color = red, width = 0.025) :
  y := arrow(e2, color = green, width = 0.025) :
  z := arrow(e3, color = blue, width = 0.025) :
  display([x, y, z], axes = normal, scaling = constrained,  tickmarks = [2, 2, 2], orientation = [30, 61]);
```



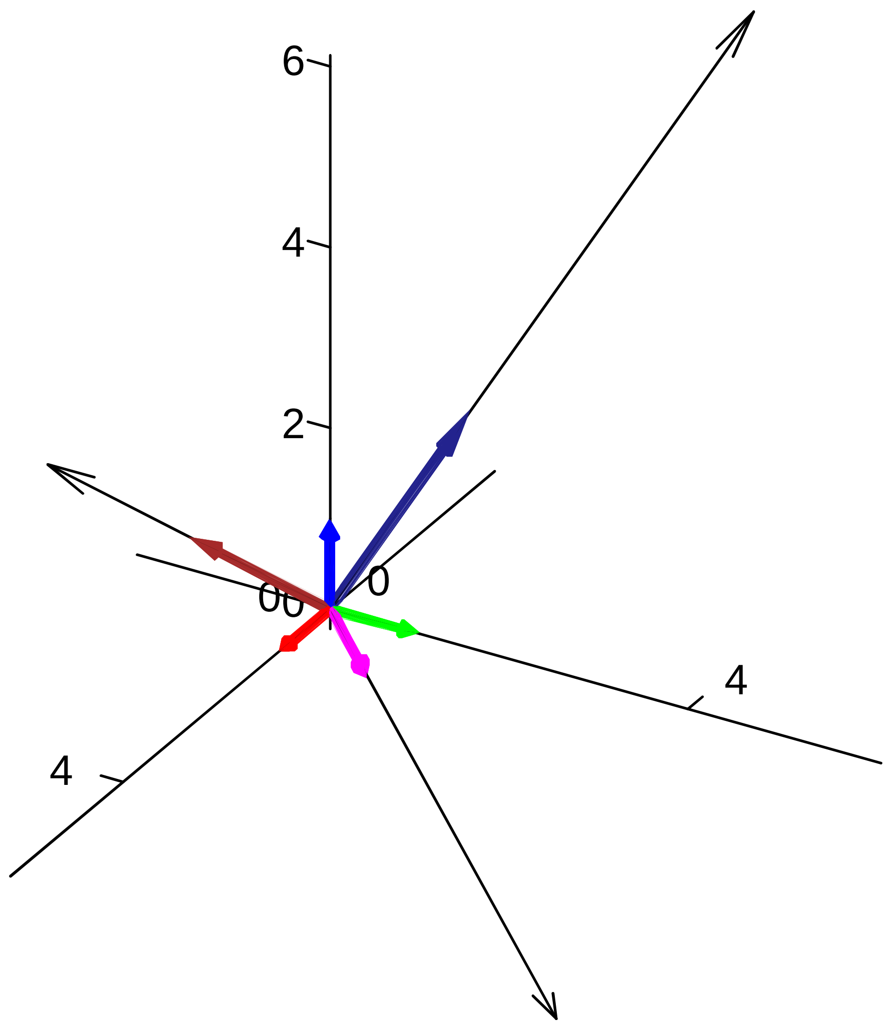
Plotting basis set 2

```
> u1 := arrow(v1, color = brown, width = 0.1) :
  u2 := arrow(v2, color = navy, width = 0.1) :
  u3 := arrow(v3, color = magenta, width = 0.1) :
  a := arrow(2· v1, color = black, width = 0.05, shape = arrow, head_length = 0.5, head_width = 0.2) :
  b := arrow(3· v2, color = black, width = 0.05, shape = arrow, head_length = 0.5, head_width = 0.2) :
  c := arrow(6· v3, color = black, width = 0.05, shape = arrow, head_length = 0.5, head_width = 0.2) :
  display([u1, u2, u3, a, b, c], axes = none, scaling = constrained, orientation = [30, 61]);
```



Plotting both bases

```
> x := arrow(e1,  color = red, width = 0.1) :
  y := arrow(e2, color = green, width = 0.1) :
  z := arrow(e3, color = blue, width = 0.1) :
  display([x, y, z, u1, u2, u3, a, b, c], axes = normal,  scaling = constrained,  tickmarks = [3, 3, 3],
                                              orientation = [30, 61]);
```



Defining vector V. V represents a vector in 3D space in terms of basis set 2

$$\mathbf{V} := 2\mathbf{v}_1 + 3\mathbf{v}_2 + 6\mathbf{v}_3$$

Writing the matrix form of vector V in terms of basis set 2

```
> V := Vector([2, 3, 6]);
```

$$V := \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \quad \textbf{(1)}$$

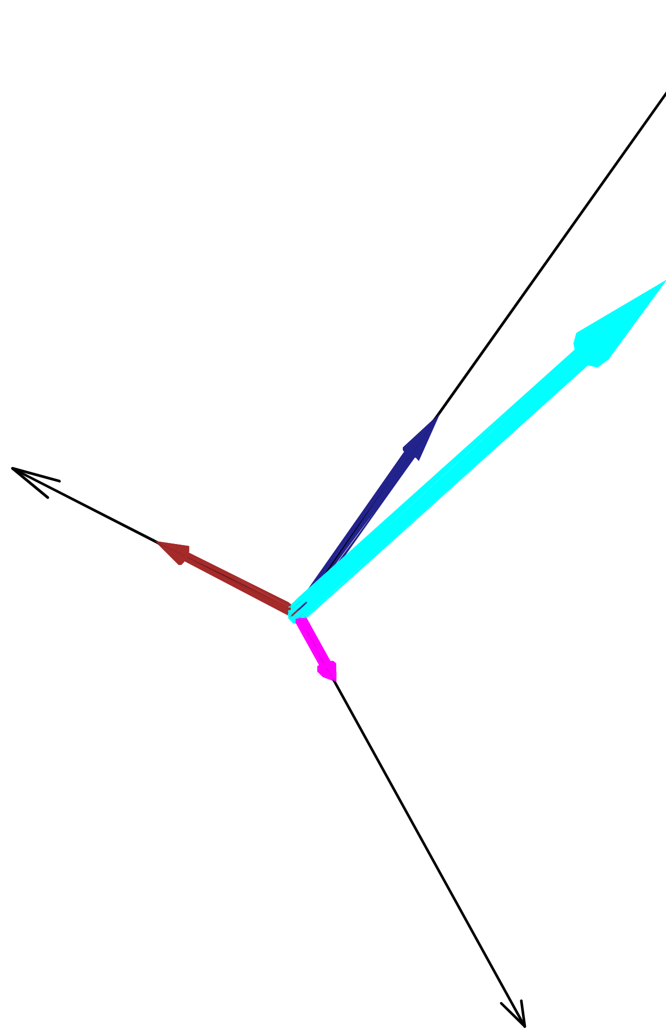
Writing Vector V in terms of basis set 1

> $R := 2\cdot \mathrm{v1} + 3\cdot \mathrm{v2} + 6\cdot \mathrm{v3};$

$$R := \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} \qquad (2)$$

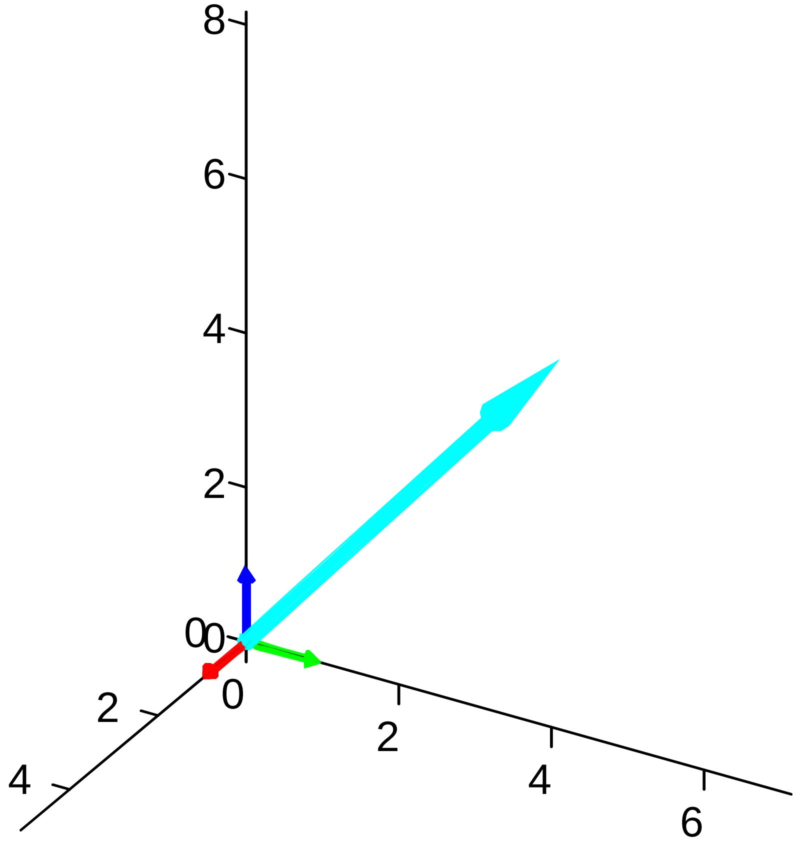
Plotting vector V, in terms of basis set 2

> $r := arrow(R, color = cyan, width = 0.20):$
> display([u1, u2, u3, a, b, c, r], axes = *none*, scaling = constrained, *orientation* = [30, 61]);



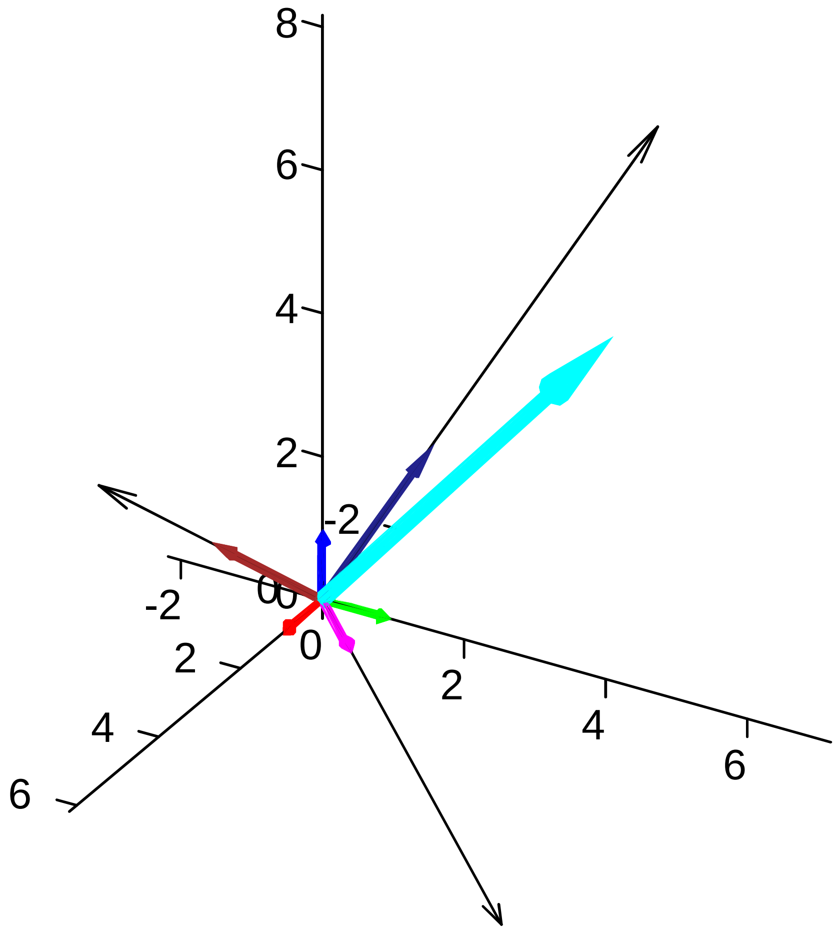
Plotting vector R, in terms of basis set 1

> display([x, y, z, *r*], axes = *normal*, scaling = constrained, *tickmarks* = [3, 3, 4], *orientation* = [30, 61]);



Plotting using both of basis sets

> display([x, y, z, u1, u2, u3, *a*, *b*, *c*, *r*], axes = *normal*, scaling = constrained, *tickmarks* = [4, 4, 4], *orientation* = [30, 61]);



Writing the transformation matrix from basis set 2 to basis set 1

> $\mathbb{S}$:= Matrix([v1, v2, v3]);

$$\mathbb{S} := \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \quad (3)$$

> $R := Multiply(\mathbb{S}, V)$;

$$R := \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} \quad (4)$$

Define the matrix S as the inverse of matrix $\mathbb{S}$

> $S := \text{MatrixInverse}(\mathbb{S});$

$$S := \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \tag{5}$$

The relationship between matrix S and its inverse is the identity matrix

> $'\mathbb{S} \cdot S' = \text{Multiply}(\mathbb{S}, S);$

$$\mathbb{S}\, S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

Define the transformation matrix from basis set 1 to basis set 2 as the matrix S

> $'S' = S;$
> $V := Multiply(S, R);$

$$S = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$V := \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \tag{7}$$

Define the transformation matrix from basis set 2 to basis set 1 as the matrix $S^{-1}=\mathbb{S}$

> $'\mathbb{S}' = \mathbb{S};$
> $R := Multiply(\mathbb{S}, V);$

$$\mathbb{S} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$R := \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} \tag{8}$$

Define matrix A in basis set 1; the standard basis

> $A := \theta \rightarrow Matrix([[\cos(\theta), -\sin(\theta), 0], [\sin(\theta), \cos(\theta), 0], [0, 0, 1]])$:
> $A := A\left(\frac{\pi}{2}\right)$;

$$A := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

> $Multiply(A, MatrixInverse(A))$;

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

Define vector Q as A · R; vector Q is written in the standard basis

> $Q := Multiply(A, R)$;

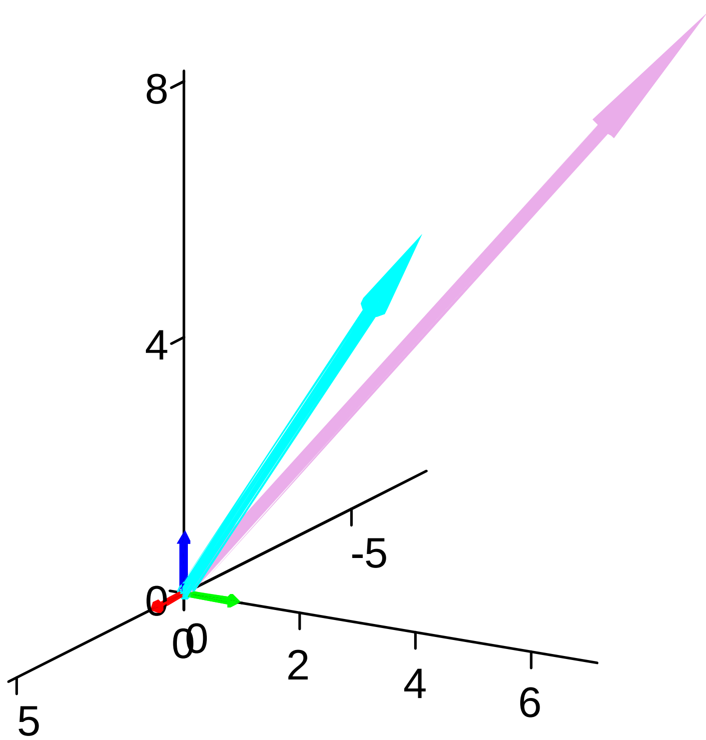
$$Q := \begin{bmatrix} -7 \\ 5 \\ 8 \end{bmatrix} \tag{11}$$

Plotting vector Q and R in the standard basis

```
> 'Q'= Q; 'R'= R;
  q := arrow(Q, color = plum, width = 0.20) :
  display([x, y, z, r, q], axes = normal,  scaling = constrained, tickmarks = [3, 3, 3],
                                    orientation = [30, 73]);
```

$$Q = \begin{bmatrix} -7 \\ 5 \\ 8 \end{bmatrix}$$

$$R = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$$



Vector V is related to vector R via the transformation matrix S.

V = S · R

Thus one can write vector U in terms of basis set 2

U = S · Q

```
> U := Multiply(S, Q);
```

$$U := \begin{bmatrix} -\frac{4}{3} \\ \frac{14}{3} \\ -1 \end{bmatrix} \tag{12}$$

Also we can say that Q = $S^{-1} \cdot U$

> $Q := Multiply(\mathbb{S}, U);$

$$Q := \begin{bmatrix} -7 \\ 5 \\ 8 \end{bmatrix} \tag{13}$$

Thus a matrix in basis set 2 is related to a matrix in basis set 1 via the transformation matrices S and S^{-1}. See Change of Basis section in the text.

$$\mathbb{A} = S \cdot A \cdot S^{-1}$$

> $\mathbb{A} := Multiply(S, Multiply(A, \mathbb{S}));$

$$\mathbb{A} := \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 1 & -1 & 0 \end{bmatrix} \tag{14}$$

> $'V' = V;$
> $U := Multiply(\mathbb{A}, V);$

$$V = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$$

$$U := \begin{bmatrix} -\frac{4}{3} \\ \frac{14}{3} \\ -1 \end{bmatrix} \tag{15}$$

> $\mathscr{A} := MatrixInverse(\mathbb{A});$

$$\mathscr{A} := \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 & 0 \end{bmatrix} \tag{16}$$

> $Multiply(\mathbb{A}, \mathscr{A});$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

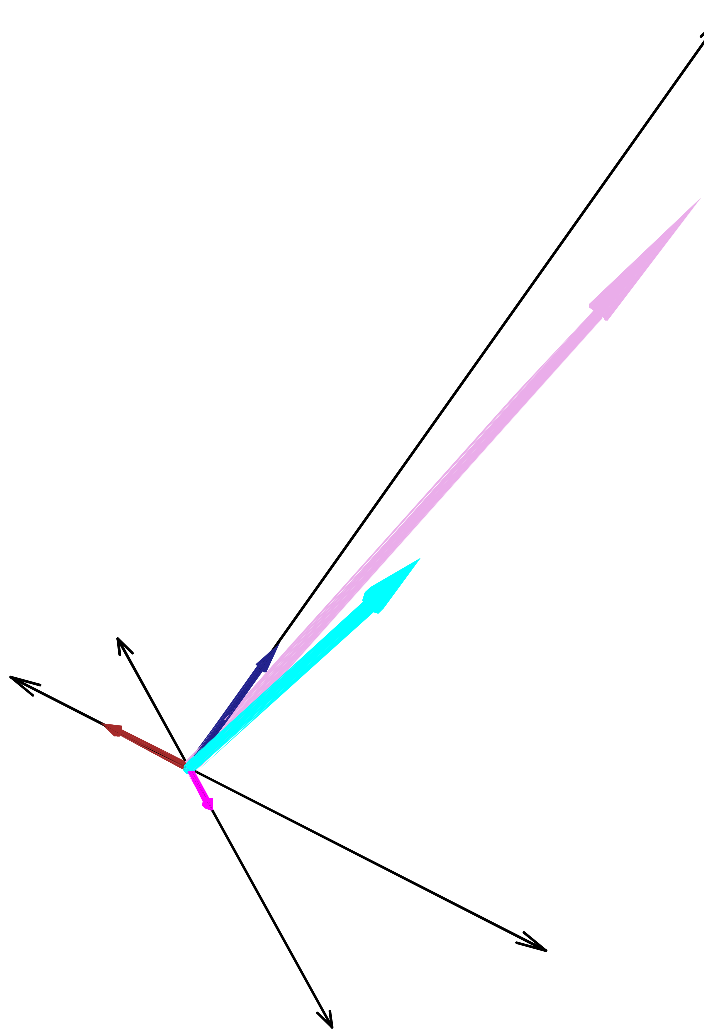
> $V := Multiply(\mathcal{A}, U);$

$$V := \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \quad (18)$$

The point here is that the matrix $\mathbb{A}$ and matrix A are related by matrices S and S^{-1}. That is, they are not inverses of each other. The inverse of $\mathbb{A}$ is $\mathcal{A}$

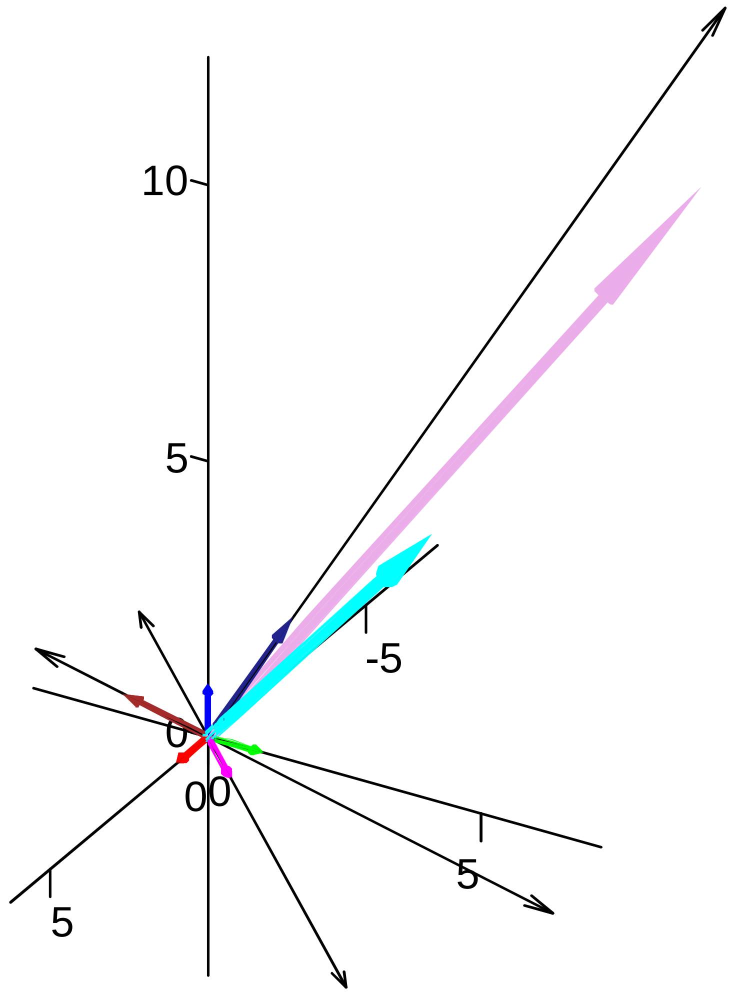
Plotting vector V and U using basis set 2

> $d :=$ arrow$(-4\cdot$ v1, color $= black$, $width = 0.05$, $shape = arrow$, $head\_length = 0.5$, $head\_width = 0.2)$:
$e :=$ arrow$(6\cdot$ v2, color $= black$, $width = 0.05$, $shape = arrow$, $head\_length = 0.5$, $head\_width = 0.2)$:
$f :=$ arrow$(-3\cdot$ v3, color $= black$, $width = 0.05$, $shape = arrow$, $head\_length = 0.5$, $head\_width = 0.2)$:
$display([u1, u2, u3, a, c, d, e, f, q, r]$, axes = none, scaling = constrained, $orientation = [30, 61]);$



Plotting both bases

> $display([x, y, z, u1, u2, u3, a, c, d, e, f, q, r], axes = normal, \ scaling = constrained, tickmarks = [3, 3, 3], orientation = [30, 61]);$



The DrawV procedure carries out the matrix vector multiplication M·C starting at θ=0 and ending at θ=π/2 using π/16 increments.

```
> DrawV := proc(C)
            local i, r, O, S, F, Z, Vzi, θ ;
            Z := [ ];
            r := evalf(1/16);   # θ = 0 → π/2 , using r increments
            for i from 0 by r to 0.5 do        # 90 degrees rotation
               θ := π· i;
               Vzi := simplify(Multiply(M(θ), C)) :   # incremental vectors
               Z := [op(Z),Vzi];       # list of vectors
            end do;
            return Z
          end proc :
```

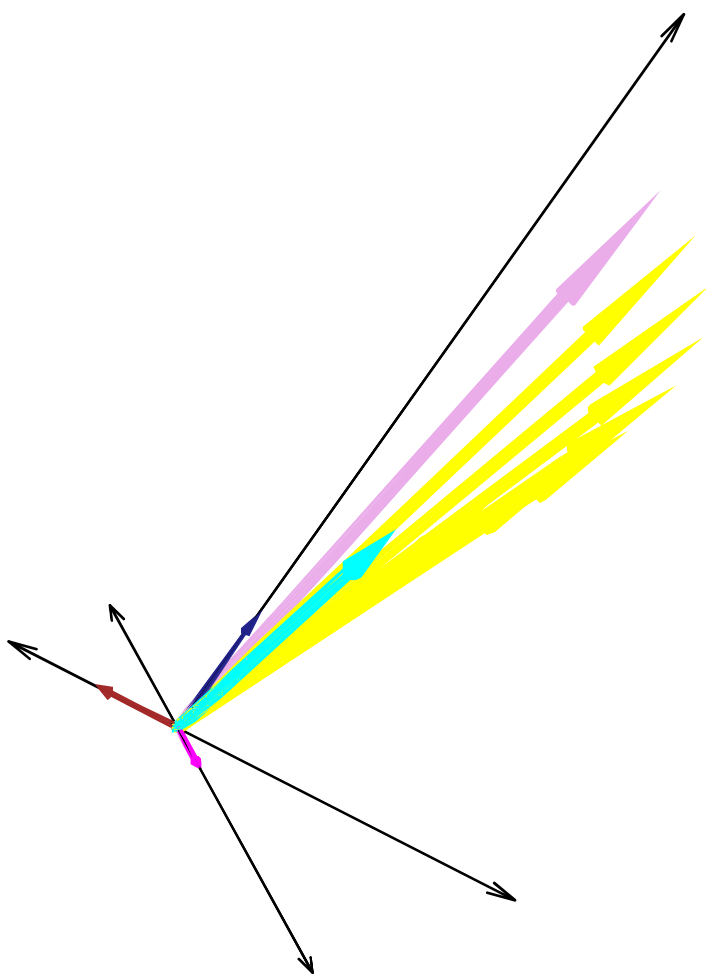
```
> M := θ → Matrix([[cos(θ), − sin(θ), 0], [sin(θ), cos(θ), 0], [0, 0, 1]]) :
  Z := DrawV(R) :
  print(The sequence of vectors is);
  print(Z);
```

The sequence of vectors is

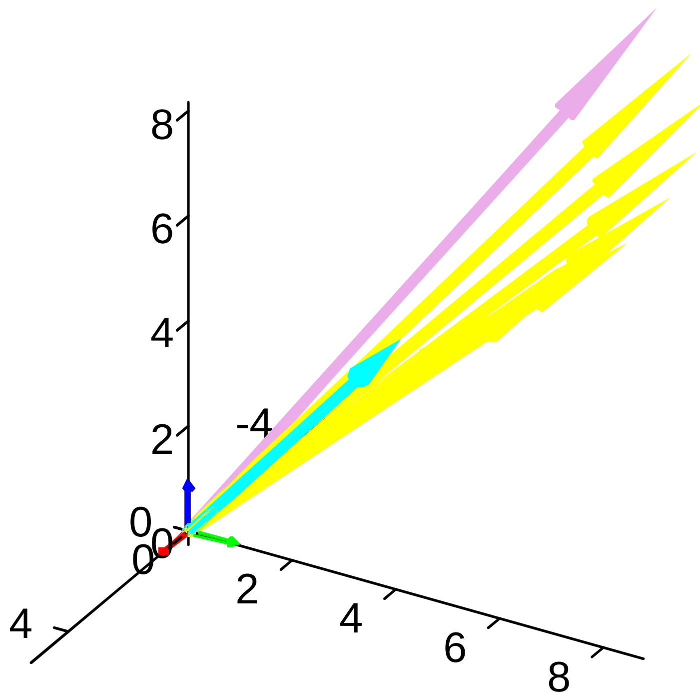
$$\left[\begin{bmatrix}5\\7\\8\end{bmatrix}, \begin{bmatrix}3.538294147\\7.840948574\\8\end{bmatrix}, \begin{bmatrix}1.940613634\\8.380573890\\8\end{bmatrix}, \begin{bmatrix}0.2683564300\\8.598138452\\8\end{bmatrix}, \begin{bmatrix}-1.414213563\\8.485281374\\8\end{bmatrix}, \begin{bmatrix}-3.042436123\\8.046339692\\8\end{bmatrix}, \begin{bmatrix}-4.553739566\\7.298181690\\8\end{bmatrix}, \begin{bmatrix}-5.890045353\\6.269558656\\8\end{bmatrix}, \begin{bmatrix}-7\\5\\8\end{bmatrix}\right] \tag{19}$$

```
> Sq := arrow(Z[2..(nops(Z) - 1)], color = yellow, width = 0.2) :      # step vectors
  Oi := arrow(Z[1], color = cyan, width = 0.20) :                     # initial vector, vector
  Fi := arrow(Z[-1], color = plum, width = 0.2) :                     # last vector in the list, vector D

  display([u1, u2, u3, a, c, d, e, f, Oi, Sq, Fi], axes = none, scaling = constrained,
                                              orientation = [30, 61],
                                              title = "Vector Rotation basis set 2 ");

  display([x, y, z, Oi, Sq, Fi], axes = normal, scaling = constrained,
                                              tickmarks = [4, 4, 4], orientation = [30, 61],
                                              title = "Vector Rotation in the Standard basis");

  display([x, y, z, u1, u2, u3, a, c, d, e, f, Oi, Sq, Fi], axes = normal, scaling = constrained,
                                              tickmarks = [4, 4, 4], orientation = [30, 61],
                                              title = " Vector Rotation");
```

Vector Rotation basis set 2



Vector Rotation in the Standard basis



Vector Rotation