

```
> restart;
> with(plots) :
```

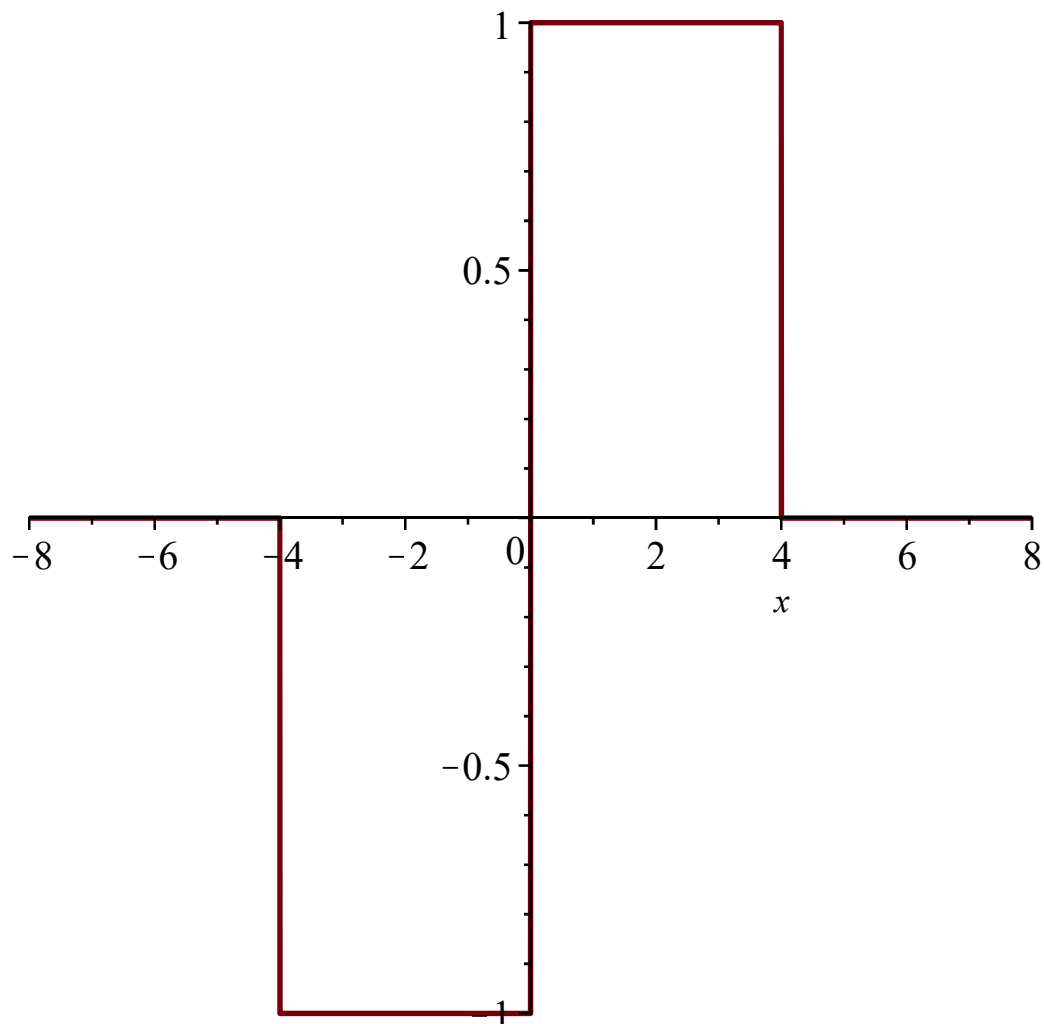
## Fourier Series Expansion

```
> f := x → piecewise( - P/2 ≤ x < 0, -1, 0 ≤ x < P/2, 1, 0 ) : 'f(x)' = f(x);
```

```
P := 8 : # period
```

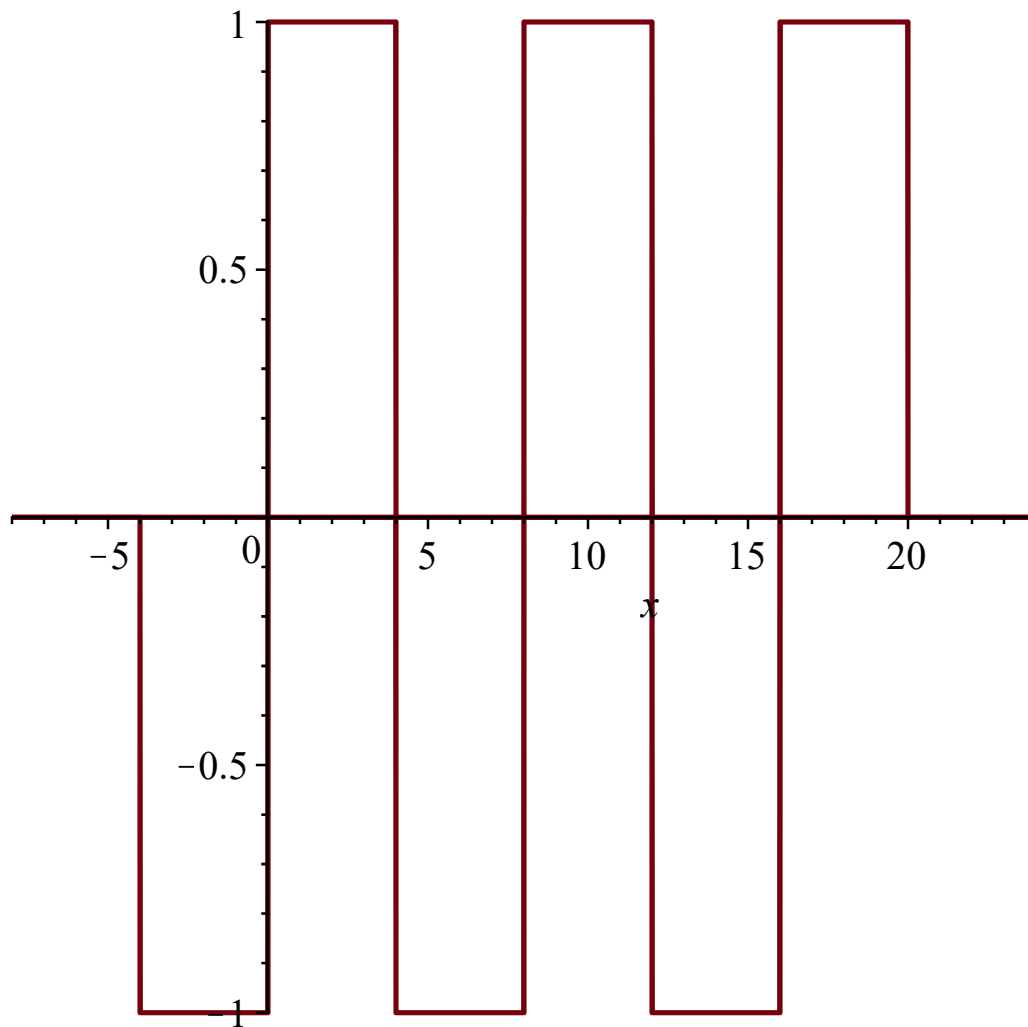
```
plot(f( x ), x = -P..P, thickness = 2);
```

$$f(x) = \begin{cases} -1 & -\frac{1}{2}P \leq x \text{ and } x < 0 \\ 1 & 0 \leq x \text{ and } x < \frac{1}{2}P \\ 0 & \text{otherwise} \end{cases}$$



### Plotting 3 periods or cycles of the function $f(x)$

```
> y := seq(plot(f(x - s), x = -P .. 3 P, thickness = 2), s = 0 .. 2 · P, P) :  
display([y]);
```



## The Fourier Series

$$> F := x \rightarrow \frac{a_0}{2} + \sum_{n=1}^N \left( a_n \cdot \cos\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) + b_n \cdot \sin\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) \right);$$

$$L := \frac{P}{2} :$$

$$F := x \rightarrow \frac{1}{2} a_0 + \sum_{n=1}^N \left( a_n \cos\left(\frac{2 n \pi x}{P}\right) + b_n \sin\left(\frac{2 n \pi x}{P}\right) \right) \quad (1)$$

$$> a_0 := \frac{2}{P} \int_{-L}^L f(x) dx; a_n := \frac{2}{P} \int_{-L}^L f(x) \cdot \cos\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) dx; b_n := \frac{2}{P} \int_{-L}^L f(x) \cdot \sin\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) dx;$$

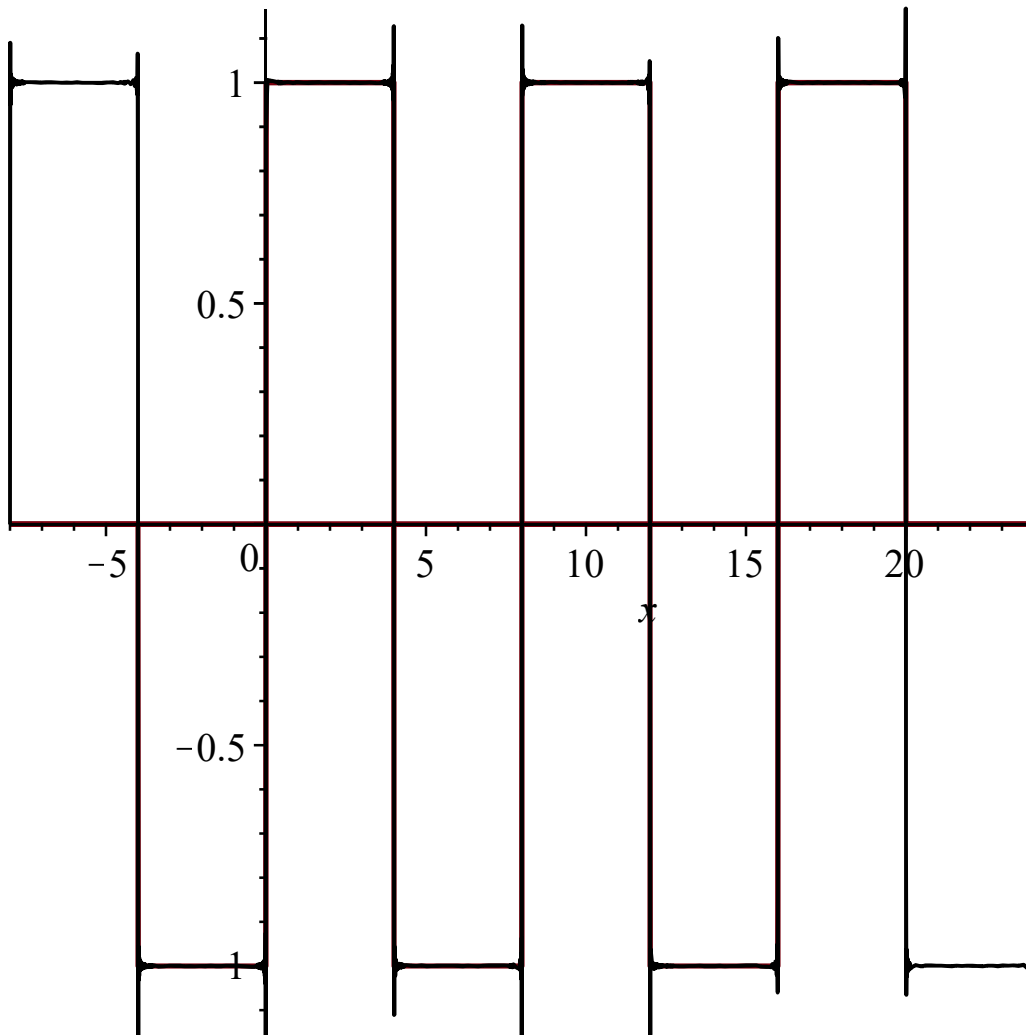
$$a_0 := 0$$

$$a_n := 0$$

$$b_n := - \frac{2 (\cos(n \pi) - 1)}{n \pi} \quad (2)$$

$$> N := 1024 : 'F(x)' = F(x) :$$

$$z := \text{plot}(F(x), x = -P..3 \cdot P, \text{color} = \text{black}) : \text{display}([y, z]);$$

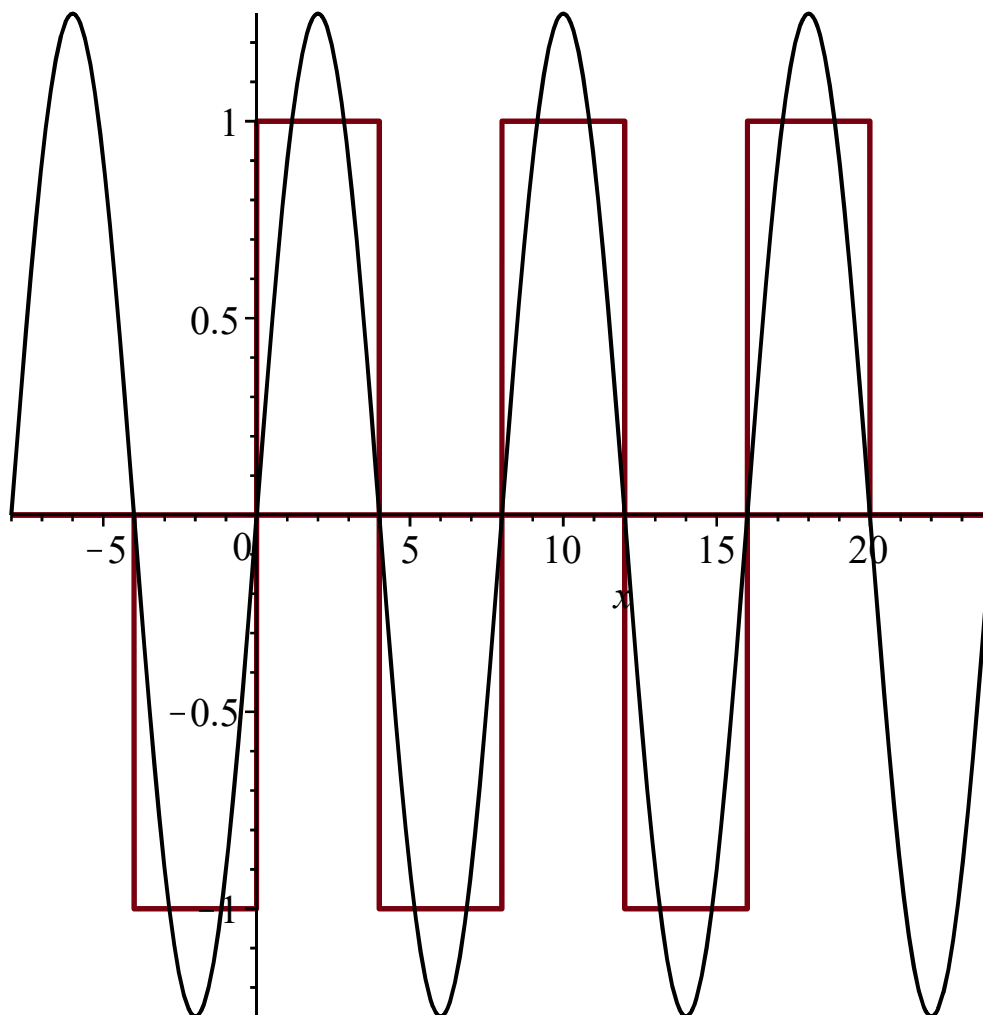


```

> for i from 1 to 10 do
  print( ); 'N'=2i; 'terms'=2i-1;
  if 2i ≤ 8 then # print the first 8 terms
    F :=  $\frac{a0}{2} + \sum_{n=1}^{2^i} an \cdot \cos\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) + \sum_{n=1}^{2^i} bn \cdot \sin\left(\frac{n \cdot 2 \pi \cdot x}{P}\right)$ ; print('F(x)'=F);
    z := plot(F, x=-P..3 P, color=black) :
  else
    F := x →  $\frac{a0}{2} + \sum_{n=1}^{2^i} an \cdot \cos\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) + \sum_{n=1}^{2^i} bn \cdot \sin\left(\frac{n \cdot 2 \pi \cdot x}{P}\right)$ ;
    z := plot(F(x), x=-P..3 P, color=black) :
  end if;
  display([y, z]);
end do;

```

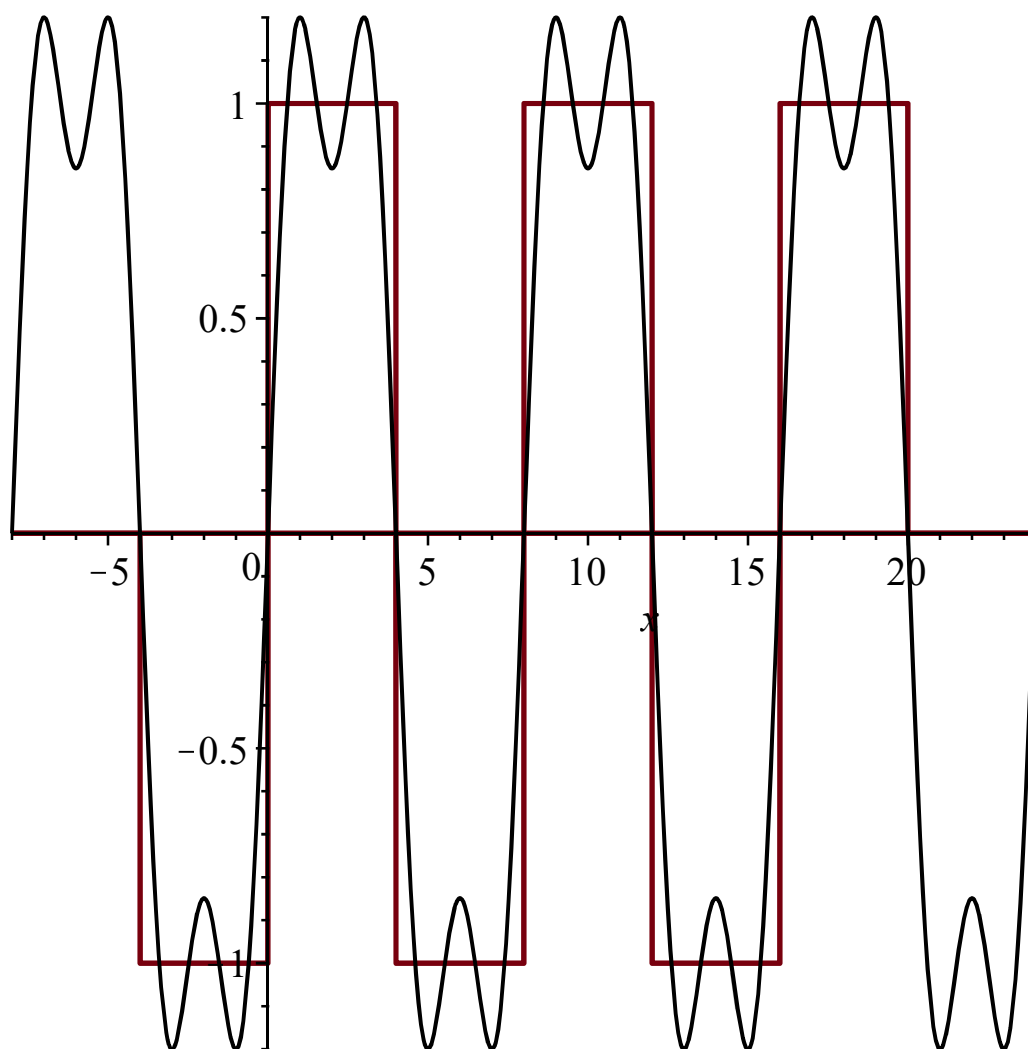
$$\begin{aligned}
 N &= 2 \\
 terms &= 1 \\
 F(x) &= \frac{4 \sin\left(\frac{1}{4} \pi x\right)}{\pi}
 \end{aligned}$$



$$N=4$$

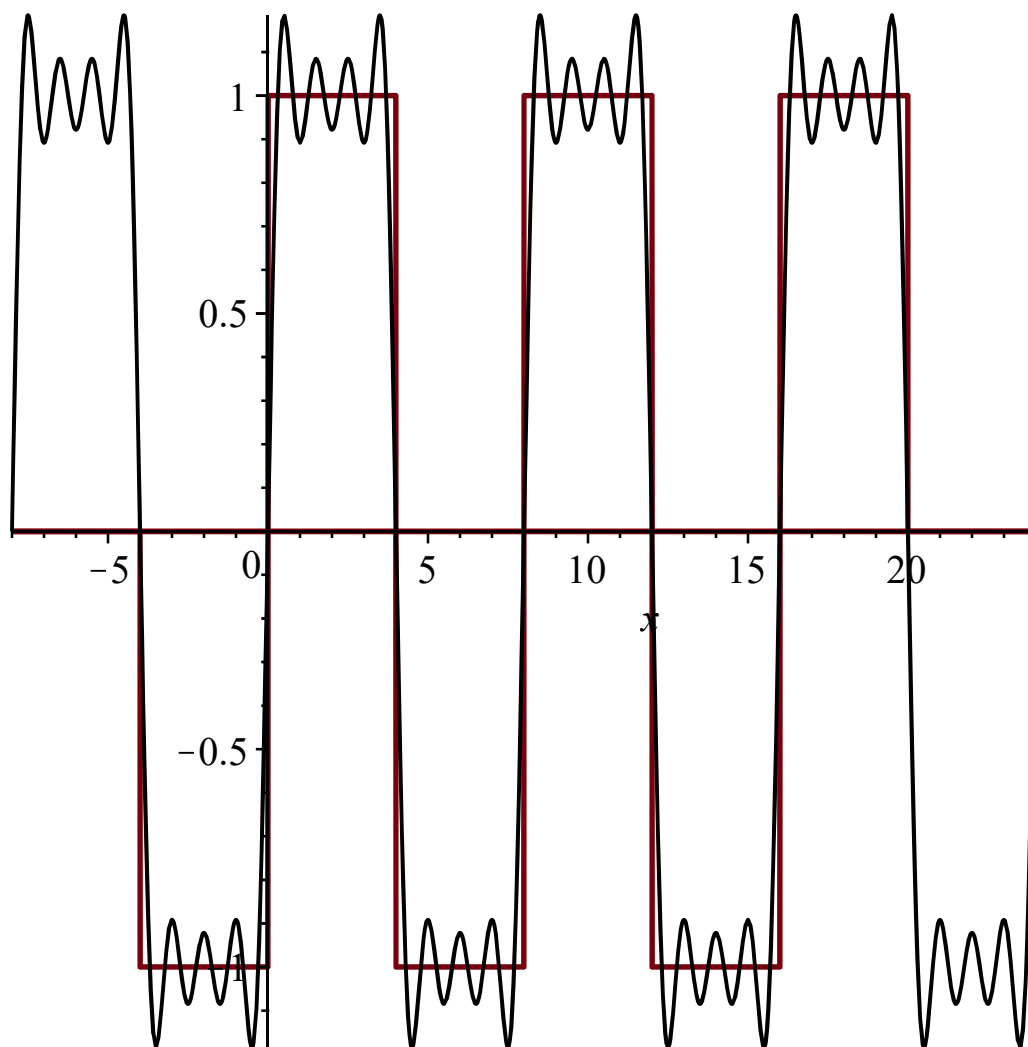
$$\text{terms}=2$$

$$F(x) = \frac{4 \sin\left(\frac{1}{4} \pi x\right)}{\pi} + \frac{4}{3} \frac{\sin\left(\frac{3}{4} \pi x\right)}{\pi}$$

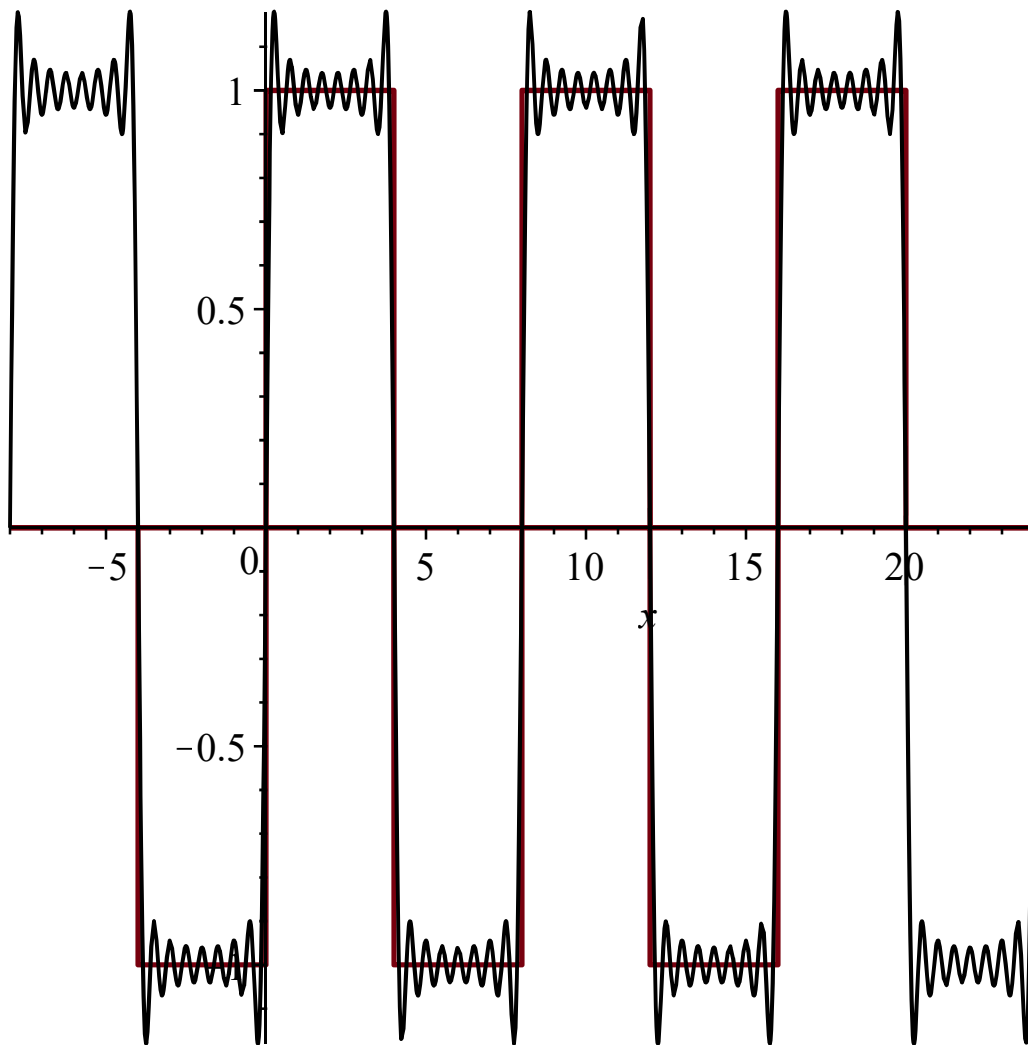


$N=8$   
 $terms=4$

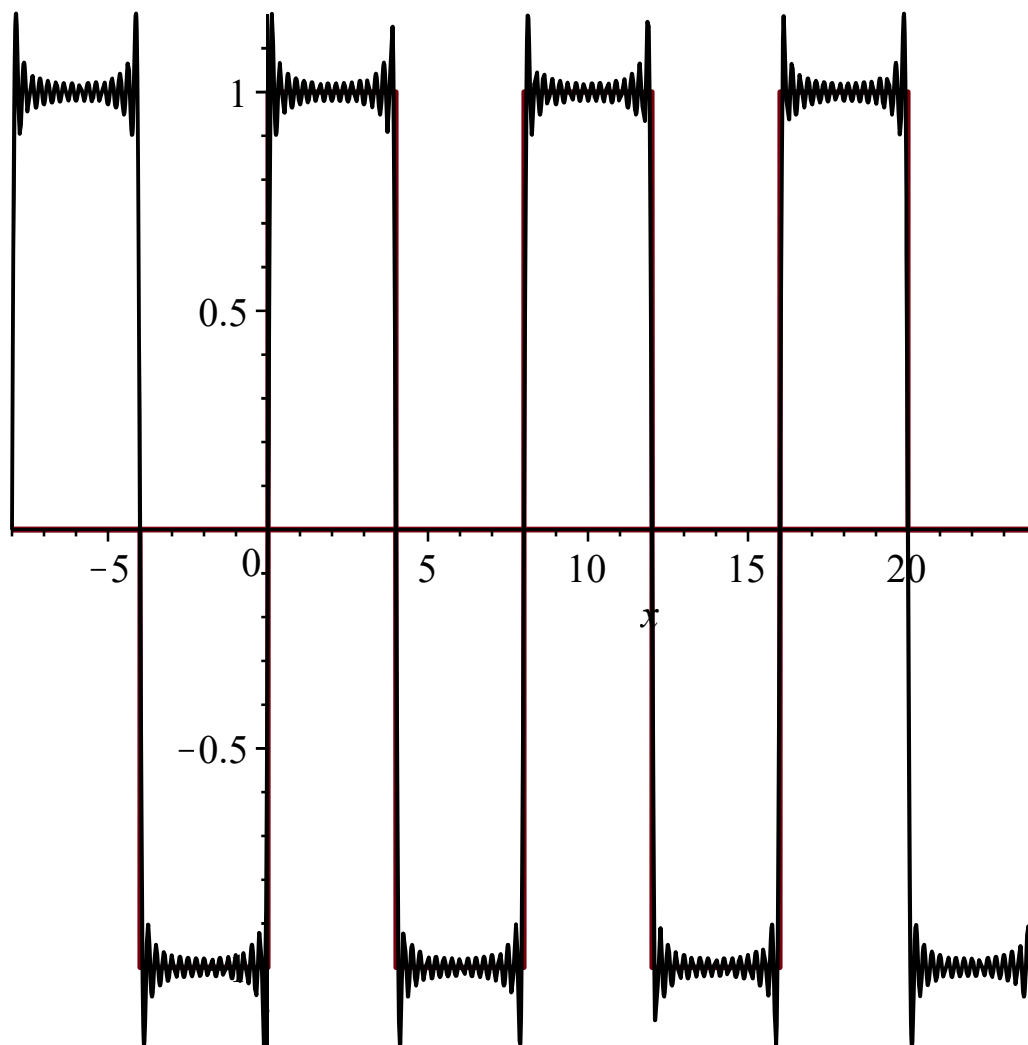
$$F(x) = \frac{4 \sin\left(\frac{1}{4} \pi x\right)}{\pi} + \frac{4}{3} \frac{\sin\left(\frac{3}{4} \pi x\right)}{\pi} + \frac{4}{5} \frac{\sin\left(\frac{5}{4} \pi x\right)}{\pi} + \frac{4}{7} \frac{\sin\left(\frac{7}{4} \pi x\right)}{\pi}$$



$N = 16$   
 $terms = 8$

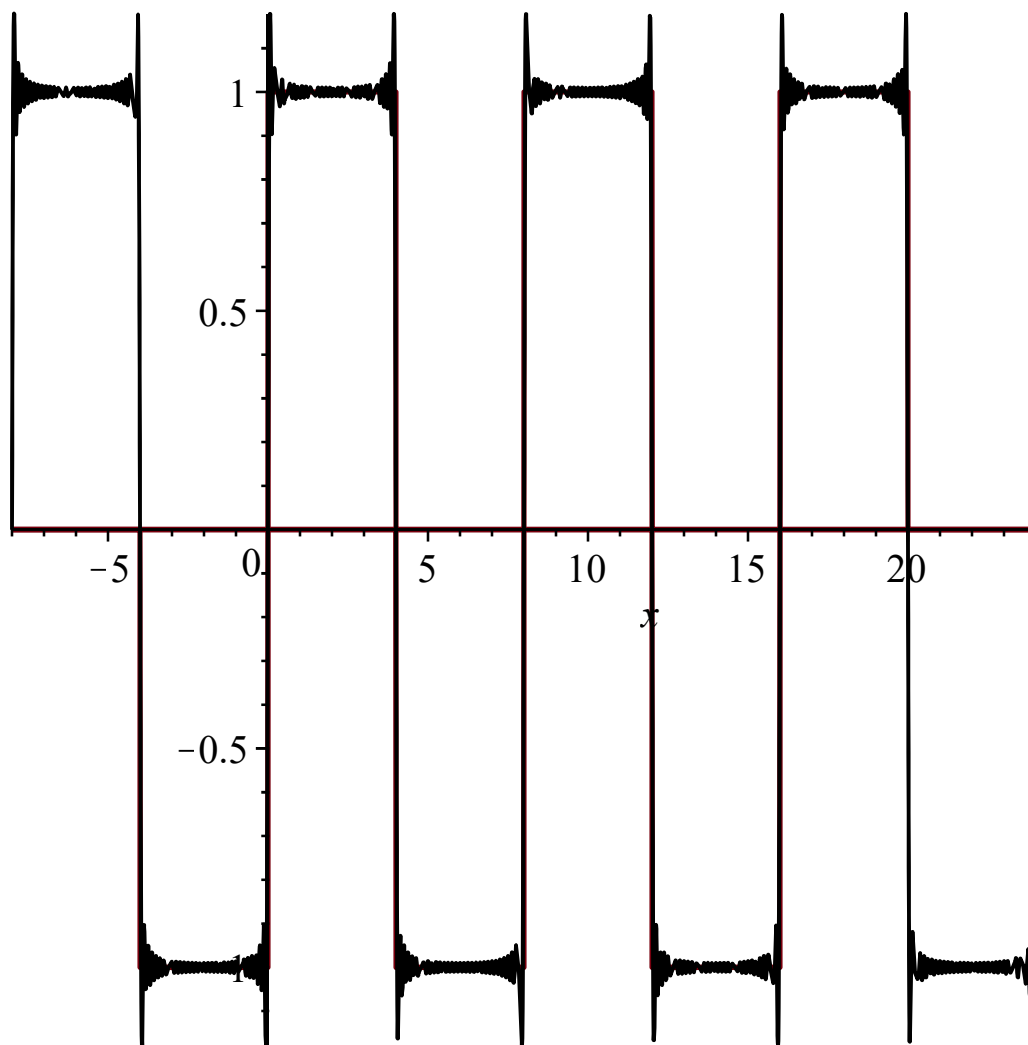


$N = 32$   
 $terms = 16$

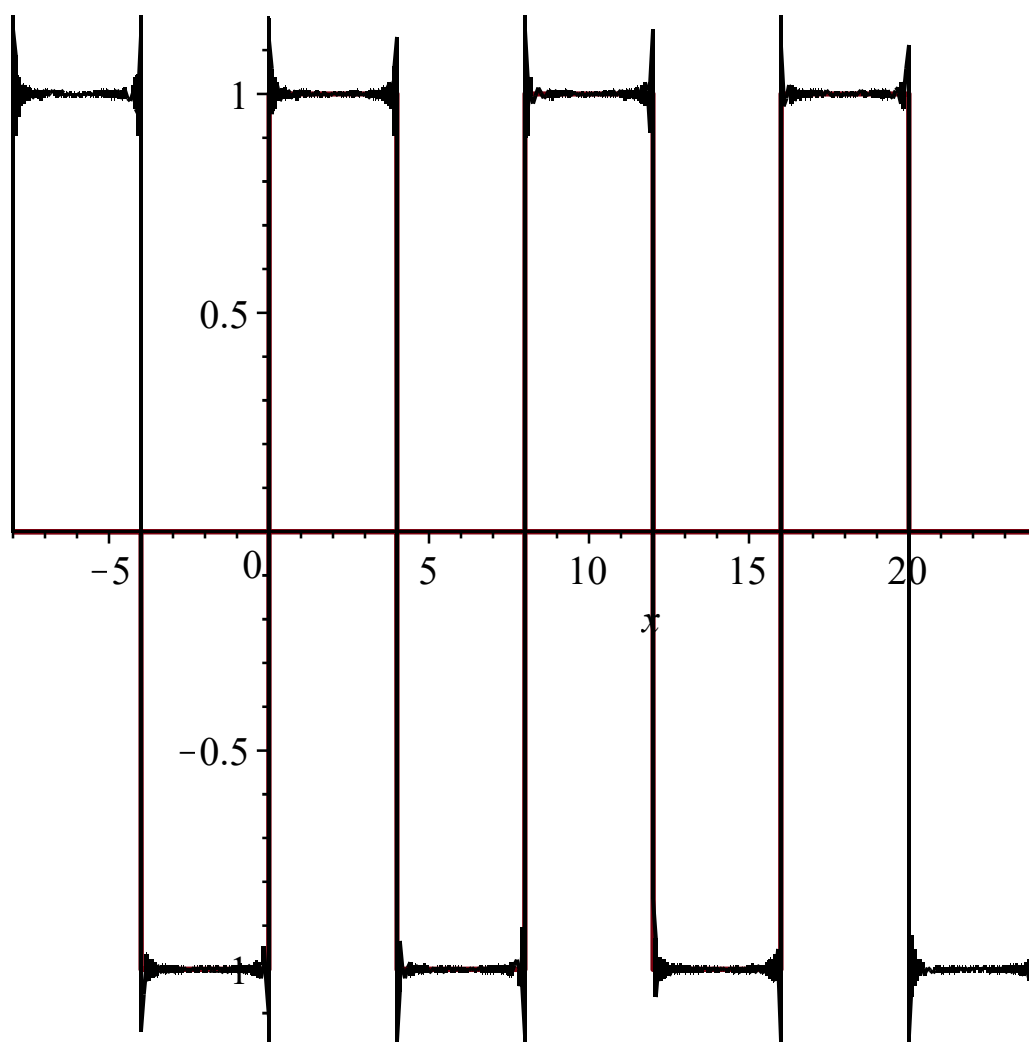




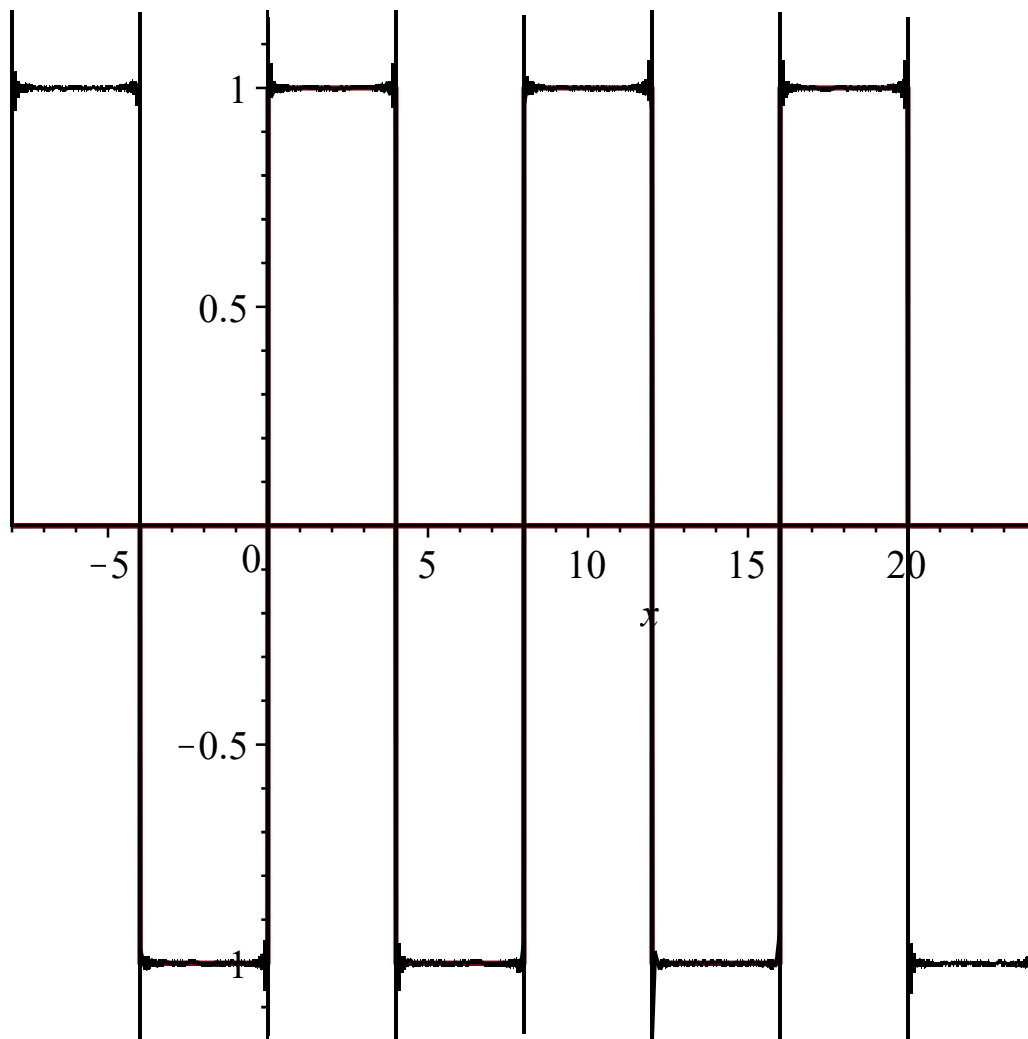
$N = 64$   
 $terms = 32$



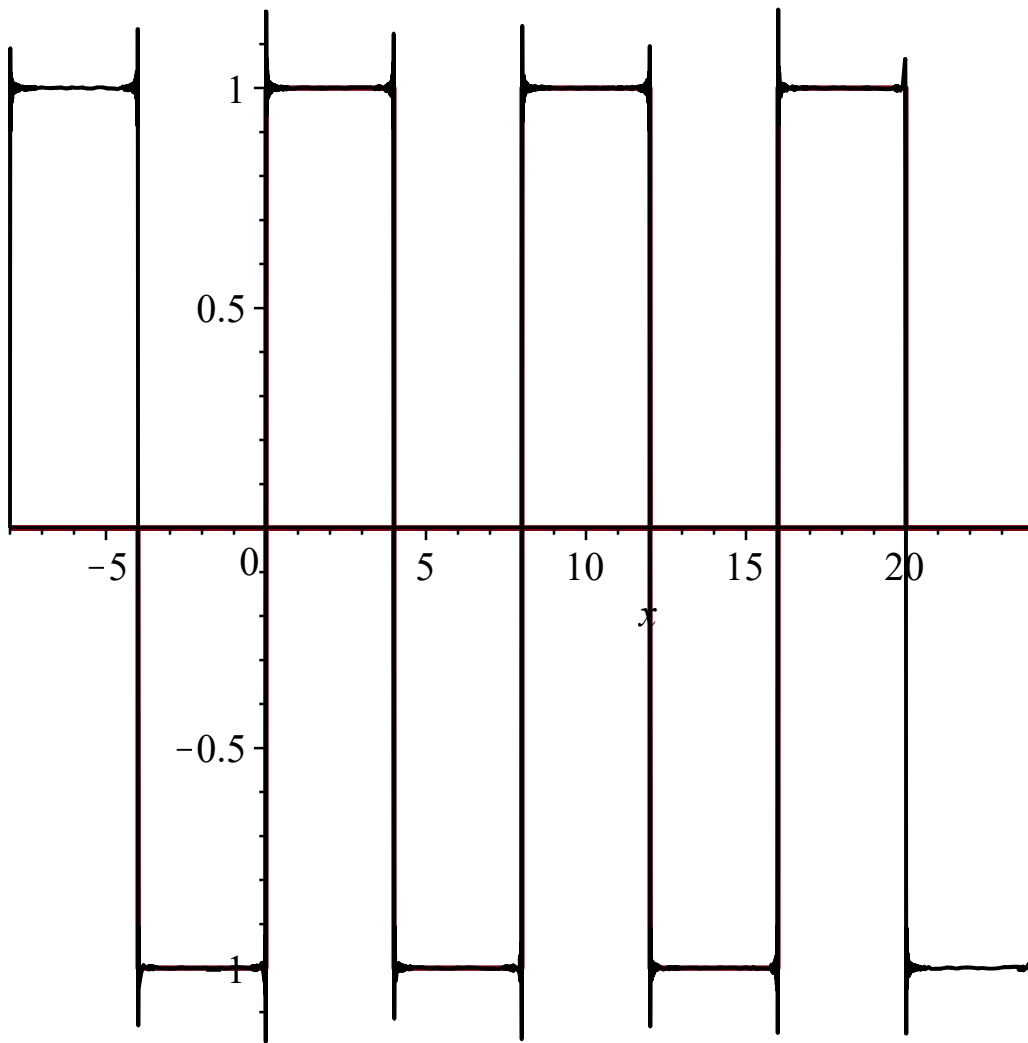
$N = 128$   
 $terms = 64$



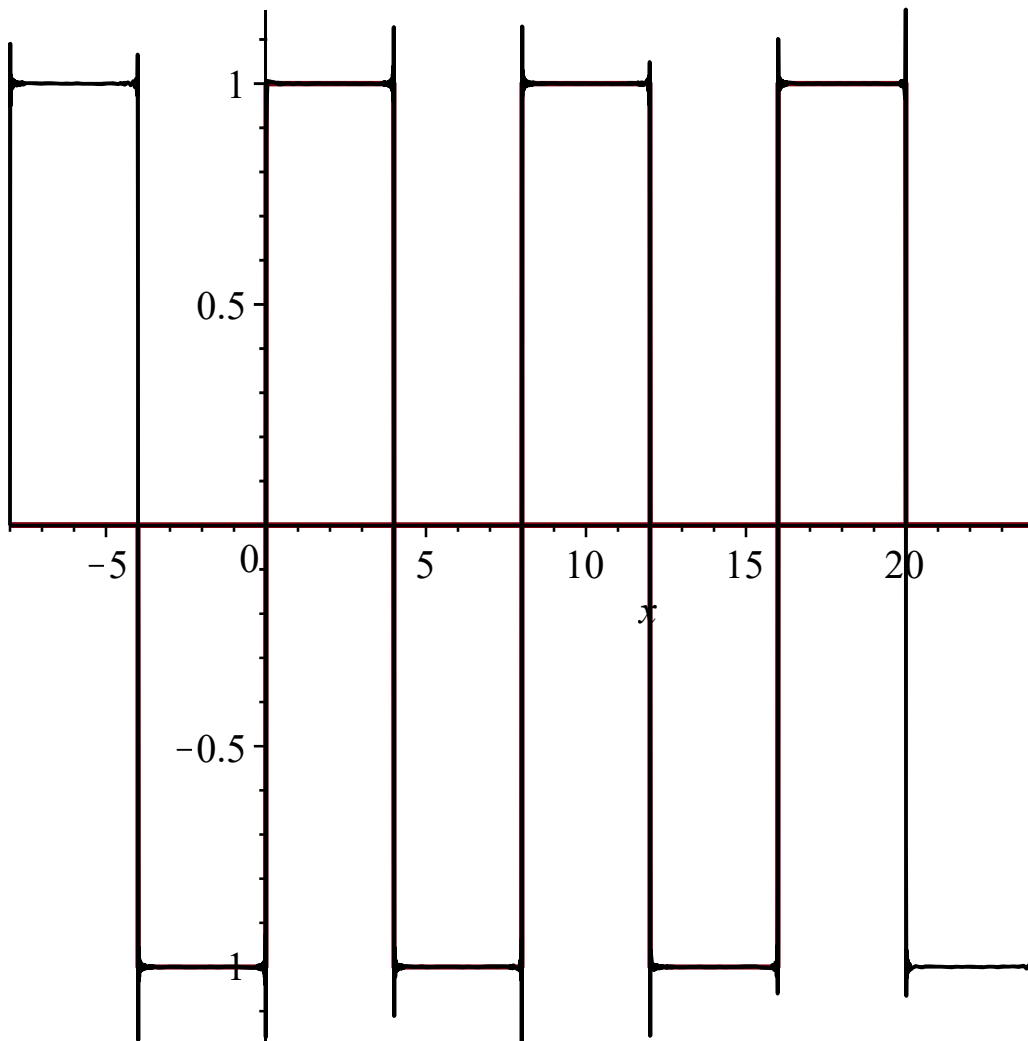
$N = 256$   
 $terms = 128$



$N=512$   
 $terms=256$



$N = 1024$   
 $terms = 512$

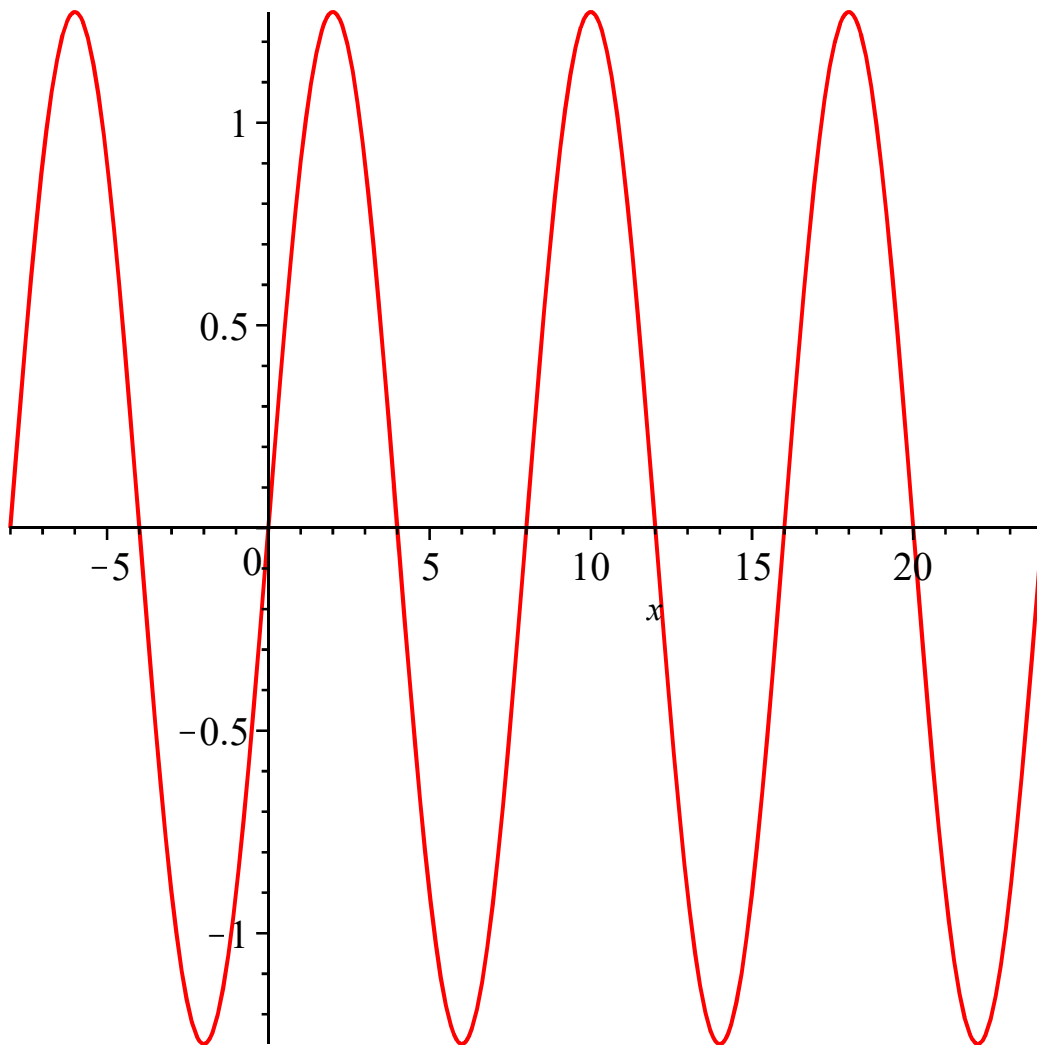


```
> N := 1 :
```

$$F1 := \frac{a0}{2} + \sum_{n=N}^N an \cdot \cos\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) + \sum_{n=N}^N bn \cdot \sin\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) : F1' = F1;$$

```
y1 := plot(F1, x = -P..3 P, color = red) :  
display([y1]);
```

$$F1 = \frac{4 \sin\left(\frac{1}{4} \pi x\right)}{\pi}$$



```

> N := 3 :
'F1'=F1;

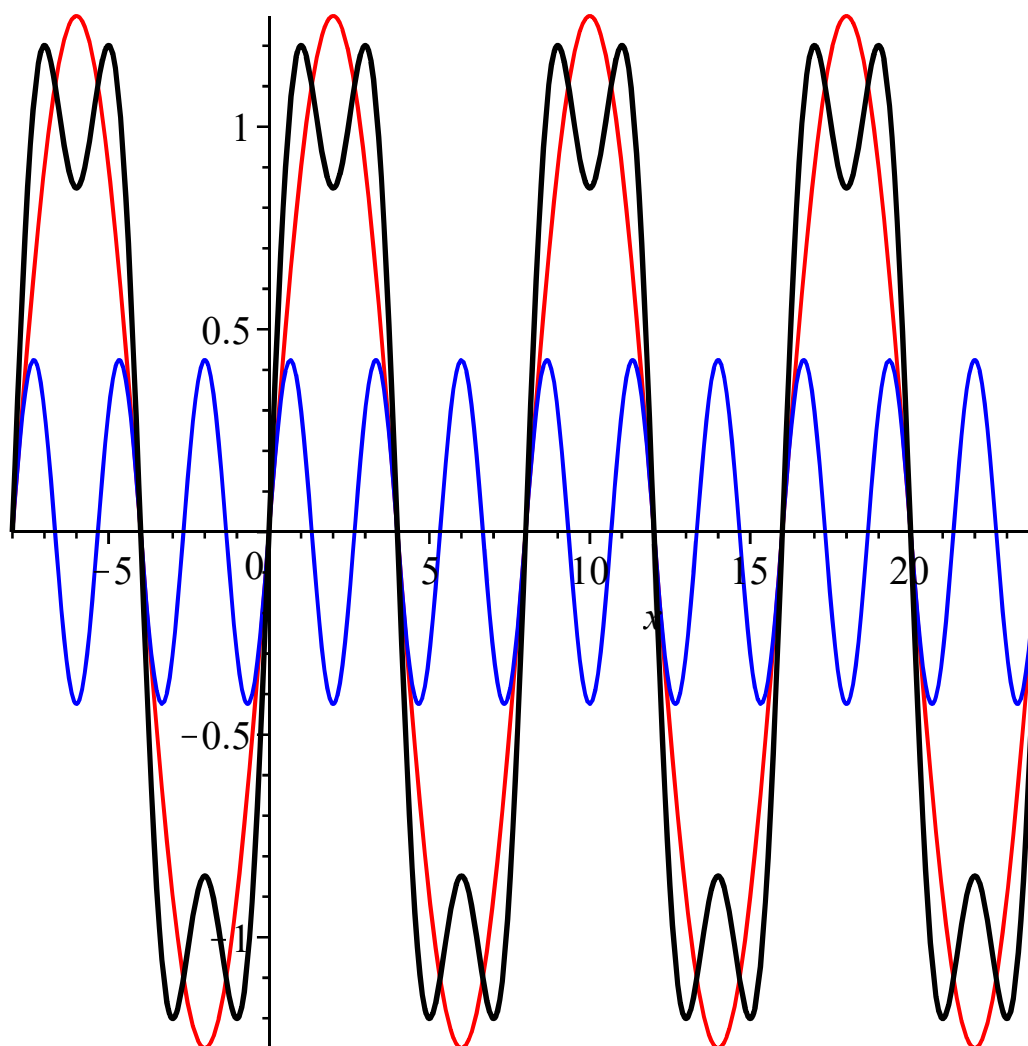
F3 :=  $\frac{a0}{2} + \sum_{n=N}^N an \cdot \cos\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) + \sum_{n=N}^N bn \cdot \sin\left(\frac{n \cdot 2 \pi \cdot x}{P}\right)$  :F3'=F3;

S1 := F1 + F3 :
y3 := plot(F3, x=-P..3 P, color=blue) :
s1 := plot(S1, x=-8..24, color=black, thickness=2) :
display([y1, y3, s1]);

```

$$F1 = \frac{4 \sin\left(\frac{1}{4} \pi x\right)}{\pi}$$

$$F3 = \frac{4}{3} \frac{\sin\left(\frac{3}{4} \pi x\right)}{\pi}$$



```

> N := 5 :
'F1'=F1;'F3'=F3;

$$F5 := \frac{a0}{2} + \sum_{n=N}^N an \cdot \cos\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) + \sum_{n=N}^N bn \cdot \sin\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) : F5' = F5;$$

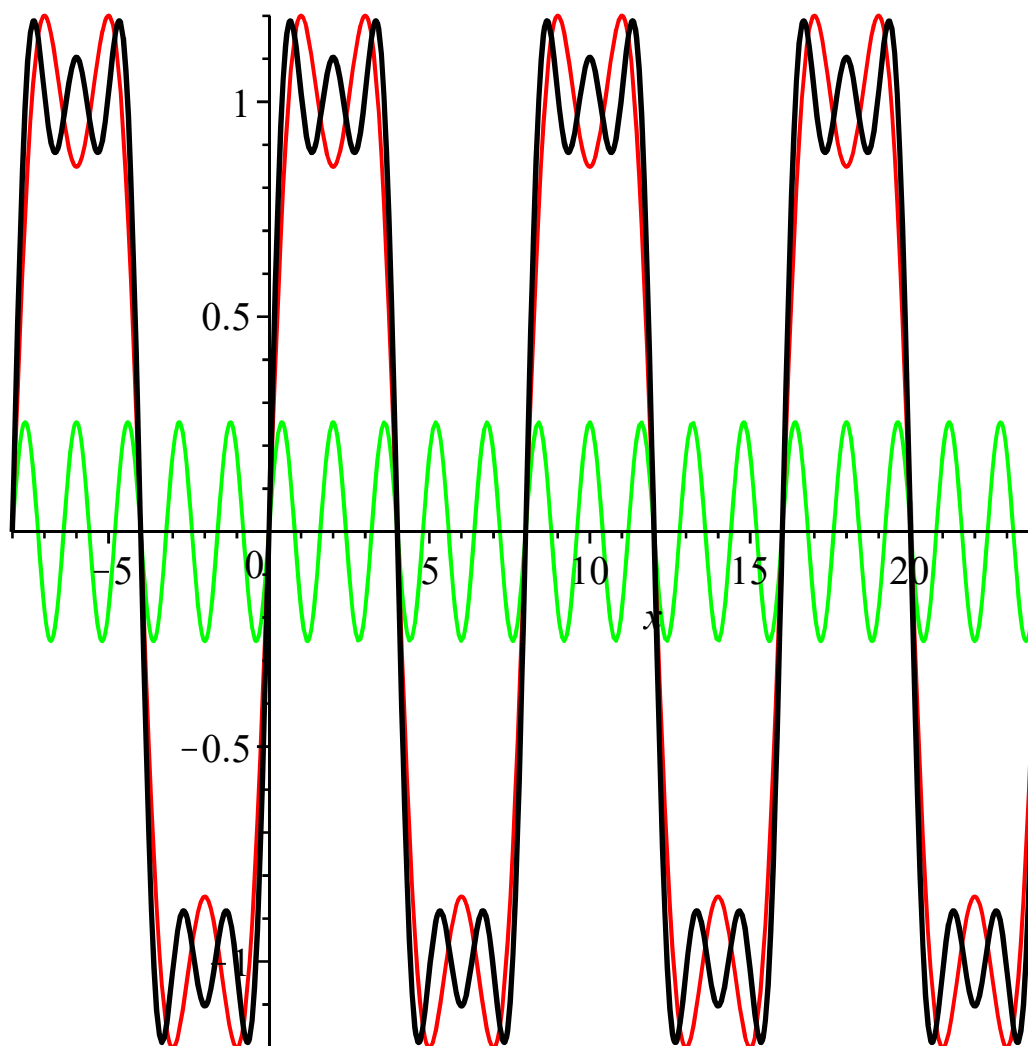
S2 := S1 + F5 :
y5 := plot(F5, x=-P..3 P, color=green) :
s1 := plot(S1, x=-8..24, color=red) :
s2 := plot(S2, x=-P..3 P, color=black, thickness=2) :
display([y5, s1, s2]);

```

$$F1 = \frac{4 \sin\left(\frac{1}{4} \pi x\right)}{\pi}$$

$$F3 = \frac{4}{3} \frac{\sin\left(\frac{3}{4} \pi x\right)}{\pi}$$

$$F5 = \frac{4}{5} \frac{\sin\left(\frac{5}{4} \pi x\right)}{\pi}$$





```

> N := 7 :
  'F1'=F1;'F3'=F3;'F5'=F5;

  F7 :=  $\frac{a0}{2} + \sum_{n=N}^N an \cdot \cos\left(\frac{n \cdot 2 \pi \cdot x}{P}\right) + \sum_{n=N}^N bn \cdot \sin\left(\frac{n \cdot 2 \pi \cdot x}{P}\right)$  : 'F7'=F7; S3 := S2 + F7 :
  y4 := plot(F7, x=-P..3 P, color=magenta) : s2 := plot(S2, x=-P..3 P, color=red) :
  s3 := plot(S3, x=-P..3 P, color=black, thickness=2) : display([y4, s2, s3]);

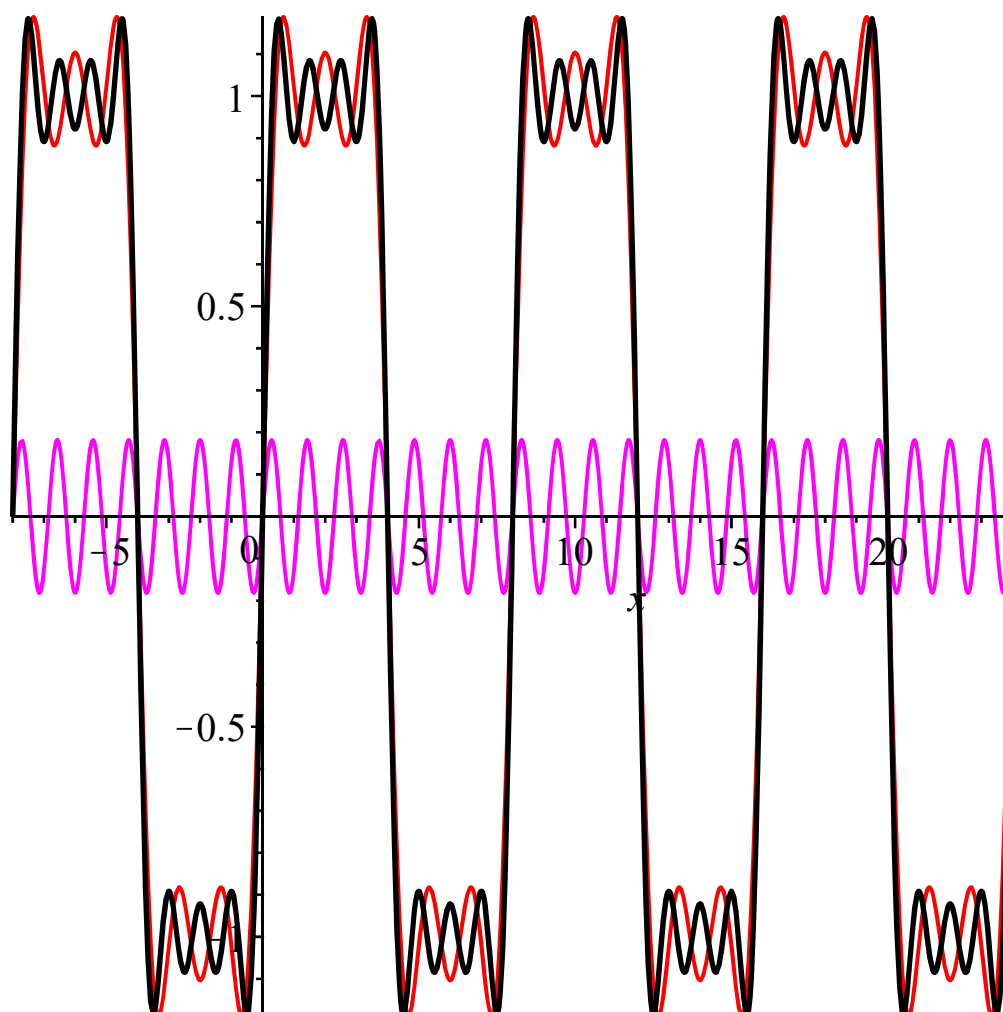
```

$$F1 = \frac{4 \sin\left(\frac{1}{4} \pi x\right)}{\pi}$$

$$F3 = \frac{4}{3} \frac{\sin\left(\frac{3}{4} \pi x\right)}{\pi}$$

$$F5 = \frac{4}{5} \frac{\sin\left(\frac{5}{4} \pi x\right)}{\pi}$$

$$F7 = \frac{4}{7} \frac{\sin\left(\frac{7}{4} \pi x\right)}{\pi}$$



```

>
y1 := plot(F1, x=-8..24, color=red) :
y2 := plot(F3, x=-8..24, color=blue) :
y3 := plot(F5, x=-8..24, color=navy) :
y4 := plot(F7, x=-8..24, color=magenta) :
s := plot(F1 + F3 + F5 + F7, x=-P..3 P, color=black, thickness=2) :
display([y1, y2, y3, y4, s]);

```

